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Summer Term 2017
30.05.2017

Bifurcation Theory

Solutions to Problem Sheet 6

Problem 16 (The Energy Method revisited)

We return to Remark III.11 (b) and Theorem II.3.

Let $T > 0$, $j \in \mathbb{N}_0$ and $g \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ as in Theorem II.3. For $n \in \mathbb{N}$, consider solutions $u_n \in C^2([0,T])$ and $\lambda_n \in \mathbb{R}$ of

$$\begin{cases} -u_n'' = g(u_n, \lambda_n) & \text{in } (0,T), \\ u_n(0) = u_n(T) = 0 \end{cases} \tag{1}$$

where u_n is j-nodal, $u_n > 0$ on $\left(0, \frac{T}{j+1}\right)$ and $\|u_n\|_{C^2([0,T])} \to 0$, $\lambda_n \to \lambda_j^*$.

Prove that $(v_n)_{n\in\mathbb{N}}$ where $v_n := \frac{u_n}{\|u_n\|_{C^2([0,T])}}$ converges in $C^2([0,T])$ and determine the limit. You can proceed as follows:

(a) Prove that there exists a constant $C > 0$ with $\|v_n\|_{C^3([0,T])} \leq C$ for every $n \in \mathbb{N}$. Conclude that the sets $\mathcal{K}_j := \left\{ v_n^{(j)} : n \in \mathbb{N} \right\} \subseteq C([0,T])$, $j = 0, 1, 2$, are bounded and equicontinuous.

(b) Deduce that $(v_n)_{n\in\mathbb{N}}$ has a $C^2([0,T])$-convergent subsequence with limit $\varphi_j \in C^2([0,T])$.

Hint: Use part (a) and the Theorem of Arzelà-Ascoli.

(c) To prove the assertion, derive the boundary value problem which is solved by φ_j.

Solution

(a) Let us first note that, for $n \in \mathbb{N}$, $v_n \in C^2([0,T])$ solves the boundary value problem

$$(\diamondsuit) \qquad \begin{cases} -v_n'' = f_n & \text{in } (0,T), \\ v_n(0) = v_n(T) = 0 \end{cases}$$

where we set $f_n(x) := \frac{1}{\|u_n\|_{C^2([0,T])}} g(\|u_n\|_{C^2([0,T])}\, v_n(x), \lambda_n)$ for $0 \le x \le T$. Then by the Chain Rule, $f_n \in C^1([0,T])$ with

$$f_n'(x) := v_n'(x)\, g_z(\|u_n\|_{C^2([0,T])}\, v_n(x), \lambda_n) = v_n'(x)\, g_z(u_n(x), \lambda_n), \qquad 0 \le x \le T,$$

and ($\diamondsuit$) implies that $v_n \in C^3([0,T])$. Since $\|u_n\|_\infty \to 0$ and $\lambda_n \to \lambda_j'$ as $n \to \infty$, we find a compact set $K \subseteq \mathbb{R} \times \mathbb{R}$ with $(u_n(x), \lambda_n) \in K$ for all $x \in [0,T]$ and $n \in \mathbb{N}$. Continuity of g_z implies that

$$C_0 := \sup_{(z,\lambda)\in K} |g_z(z,\lambda)| < \infty.$$

We now let $C := 3 + C_0$ and show that, for every $n \in \mathbb{N}$, $\|v_n\|_{C^3([0,T])} \le C$. First, we recall that $\|v_n\|_{C^3([0,T])} = \|v_n\|_\infty + \|v_n'\|_\infty + \|v_n''\|_\infty + \|v_n'''\|_\infty$ and note that, by definition of v_n,

$$|v_n(x)| \le 1, \quad |v_n'(x)| \le 1, \quad |v_n''(x)| \le 1 \qquad \text{for all } x \in [0,T].$$

Moreover, by ($\diamondsuit$), we have for every $x \in (0,T)$

$$|v_n'''(x)| = |f_n'(x)| = |v_n'(x)\, g_z(u_n(x), \lambda_n)| \le 1 \cdot C_0 = C_0.$$

We conclude that $\|v_n\|_\infty \le 1$, $\|v_n'\|_\infty \le 1$, $\|v_n''\|_\infty \le 1$ and $\|v_n'''\|_\infty \le C_0$, hence $\|v_n\|_{C^3([0,T])} \le 3 + C_0 = C$, as asserted.

Now consider $j \in \{0,1,2\}$.

Boundedness of $\mathcal{K}_j$: Since, for $n \in \mathbb{N}$, $\left\|v_n^{(j)}\right\|_\infty \le 1$, $\mathcal{K}_j$ is a bounded subset of $C([0,T])$.

Equicontinuity of $\mathcal{K}_j$: Here we exploit that $\left\|v_n^{(j+1)}\right\|_\infty \le C$ for every $n \in \mathbb{N}$. More precisely, we let $\varepsilon > 0$ and define $\delta := \frac{\varepsilon}{C}$. Then, for every $n \in \mathbb{N}$ and $x, y \in [0,T]$, we have the implication

$$|x-y| < \delta \quad \Rightarrow \quad |v_n^{(j)}(x) - v_n^{(j)}(y)| = \left| \int_x^y v_n^{(j+1)}(t)\, \mathrm{d}t \right| \le |x-y| \cdot C < \varepsilon,$$

which proves (both uniform Lipschitz continuity and) equicontinuity of the family $\mathcal{K}_j$.

(b) By part (a), the set $\mathcal{K}_2$ is bounded and equicontinuous. The Theorem of Arzelà-Ascoli thus yields $\phi_2 \in C([0,T])$ and a subsequence with $v_{n_k}'' \to \phi_2$ uniformly as $k \to \infty$.

We now consider $\mathcal{K}_1' := \left\{v_{n_k}' : k \in \mathbb{N}\right\}$, which is also bounded and equicontinuous; similarly, we find $\phi_1 \in C([0,T])$ and a sub-subsequence with $v_{n_{k_l}}' \to \phi_1$ uniformly as $l \to \infty$.

Finally, we let $\mathcal{K}_0' := \left\{v_{n_{k_l}}' : l \in \mathbb{N}\right\}$ and identify $\varphi_j \in C([0,T])$ and another subsequence with $v_{n_{k_{l_m}}} \to \varphi_j$ uniformly as $m \to \infty$.

For convenience, we denote $(v_{n_{k_{l_m}}})_{m\in\mathbb{N}}$ again by $(v_{n_m})_{m\in\mathbb{N}}$. We then have, as $m \to \infty$,

$$v_{n_m} \to \varphi_j, \quad v_{n_m}' \to \phi_1, \quad v_{n_m}'' \to \phi_2 \qquad \text{uniformly on } [0,T].$$

Due to uniform convergence of the derivatives, we deduce that $\varphi_j \in C^2([0,T])$ with $\varphi_j' = \phi_1$ and $\varphi_j'' = \phi_2$, which implies that $v_{n_m} \to \varphi_j$ in $C^2([0,T])$, as asserted.

(c) We consider the subsequence $(v_{n_m})_{m\in\mathbb{N}}$ from part (b) and intend to pass to the limit $m \to \infty$ in problem $(\diamondsuit)$. For the boundary values, we exploit that uniform convergence implies pointwise convergence, hence

$$\varphi_j(0) = \lim_{m\to\infty} v_{n_m}(0) = 0, \qquad \varphi_j(T) = \lim_{m\to\infty} v_{n_m}(T) = 0.$$

Likewise, for every fixed $x \in (0,T)$, we have[1]

$$\begin{aligned} -\varphi_j''(x) &= \lim_{m\to\infty} -v_{n_m}''(x) = \lim_{m\to\infty} f_{n_m}(x) \\ &= \lim_{m\to\infty} \frac{1}{\|u_{n_m}\|_{C^2([0,T])}} g(\|u_{n_m}\|_{C^2([0,T])}\, v_{n_m}(x), \lambda_{n_m}) \\ &= \lim_{m\to\infty} \int_0^1 g_z(\tau\, \|u_{n_m}\|_{C^2([0,T])}\, v_{n_m}(x), \lambda_{n_m}) v_{n_m}(x)\, \mathrm{d}\tau \\ &= \int_0^1 \lim_{m\to\infty} g_z(\tau\, \|u_{n_m}\|_{C^2([0,T])}\, v_{n_m}(x), \lambda_{n_m}) v_{n_m}(x)\, \mathrm{d}\tau \qquad = g_z(0,\lambda_j^*)\, \varphi_j(x) \end{aligned}$$

where the interchange of limit and integration can be justified by dominated convergence, and we finally used that $\|u_{n_m}\|_{C^2([0,T])} \to 0$ as $m \to \infty$. Theorem II.3 gives, since the point $(0,\lambda_j^*)$ is assumed to be a bifurcation point, $g_z(0,\lambda_j^*) = \big((j+1)\frac{\pi}{T}\big)^2$. We summarize:

$$\begin{cases} -\varphi_j'' = \big((j+1)\frac{\pi}{T}\big)^2\ \varphi_j \quad \text{in } (0,T), \\ \varphi_j(0) = \varphi_j(T) = 0, \end{cases}$$

This implies

$$\varphi_j(x) = \alpha_j \cdot \sin\Big((j+1)\frac{\pi}{T}\cdot x\Big) \qquad (0 \le x \le T)$$

with a factor $\alpha_j > 0$ determined[2] by the normalization condition

$$1 = \lim_{m\to\infty} \|v_{n_m}\|_{C^2([0,T])} = \|\varphi_j\|_{C^2([0,T])} = \alpha_j \cdot \left(1 + (j+1)\frac{\pi}{T} + \Big((j+1)\frac{\pi}{T}\Big)^2\right).$$

Note that, so far, we have only shown the convergence of a subsequence $(v_{n_m})_{m\in\mathbb{N}}$ of $(v_n)_{n\in\mathbb{N}}$ to φ_j. However, the discussion of parts (b) and (c) applies to any (arbitrary) subsequence of $(v_n)_{n\in\mathbb{N}}$, yielding a sub-subsequence which converges in $C^2([0,T])$ to the unique limit φ_j. This implies convergence of the sequence $(v_n)_{n\in\mathbb{N}}$ as a whole. □

[1] recalling from Theorem II.3 that g is assumed to be odd in z, hence $g(0,\lambda_n) = 0$, and applying the Fundamental Theorem of Calculus

[2] Note that $\alpha_j > 0$ since we assume $v_n > 0$ on $\big(0, \frac{T}{j+1}\big)$.

Problem 17 (The Crandall-Rabinowitz Theorem in finite dimensions)

(a) Determine all bifurcation points $(0, \lambda_0) \in \mathbb{R}^2 \times \mathbb{R}$ of the nonlinear system

$$\begin{cases} \sin(x_1 + \lambda x_2) = x_1, \\ \cos(\lambda x_1 + x_2) = 1 + x_1. \end{cases} \tag{2}$$

(b) Let $A \in \mathbb{R}^{n\times n}$ be symmetric, and $J \in \mathbb{R}^{n\times n}$. For $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we study the equation

$$Ax = \lambda x + |x|^2 Jx. \tag{3}$$

Discuss the existence of nontrivial solutions in a neighborhood of $(0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ if

(i) λ_0 is not an eigenvalue of A, **(ii)** λ_0 is a simple eigenvalue of A.

Solution

(a) *Assertion:* $(0, \lambda_0)$ *is a bifurcation point of problem* (2) *if and only if* $\lambda_0 = 0$.

Solving problem (2) is equivalent to finding zeros of the function

$$F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \quad F(x_1, x_2, \lambda) := \begin{pmatrix} \sin(x_1 + \lambda x_2) - x_1 \\ \cos(\lambda x_1 + x_2) - 1 - x_1 \end{pmatrix}.$$

Then $F(0, 0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$, and F is twice continuously differentiable with

$$F_x(x_1, x_2, \lambda) = \begin{pmatrix} \cos(x_1 + \lambda x_2) - 1 & \lambda\cos(x_1 + \lambda x_2) \\ -\lambda\sin(\lambda x_1 + x_2) - 1 & -\sin(\lambda x_1 + x_2) \end{pmatrix},$$

$$F_{x\lambda}(x_1, x_2, \lambda) = \begin{pmatrix} -x_2\sin(x_1 + \lambda x_2) & \cos(x_1 + \lambda x_2) - \lambda x_2\sin(x_1 + \lambda x_2) \\ -\sin(\lambda x_1 + x_2) - \lambda x_1\cos(\lambda x_1 + x_2) & -x_1\cos(\lambda x_1 + x_2) \end{pmatrix}$$

for $x_1, x_2, \lambda \in \mathbb{R}$, and in particular, for $\lambda_0 \in \mathbb{R}$,

$$F_x(0, 0, \lambda_0) = \begin{pmatrix} 0 & \lambda_0 \\ -1 & 0 \end{pmatrix}, \qquad F_{x\lambda}(0, 0, \lambda_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If $\lambda_0 \neq 0$, we have that $\det F_x(0, 0, \lambda_0) = \lambda_0 \neq 0$; hence $F_x(0, 0, \lambda_0)$ (interpreted as a linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$) is a homeomorphism. Corollary III.9 then states that $(0, \lambda_0)$ cannot be a bifurcation point for problem (2).

Now let $\lambda_0 = 0$ and apply the Crandall-Rabinowitz Bifurcation Theorem: We have

$$\ker F_x(0, 0, 0) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \qquad \operatorname{ran} F_x(0, 0, 0) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},$$

and thus, the simplicity condition (S) holds. Moreover, with $\varphi := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have

$$F_{x\lambda}(0,0,0)[\varphi] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \text{ran } F_x(0,0,0),$$

which shows that the transversality condition (T) is also satisfied. Theorem IV.2 now states that $(0,0)$ is a bifurcation point. □

(b) Solving problem (3) is equivalent to finding zeros of the function

$$F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad F(x,\lambda) := Ax - \lambda x - |x|^2 Jx.$$

We note that $F(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$, and that F is twice continuously differentiable with

$$F_x(x,\lambda)[h] = Ah - \lambda h - |x|^2 Jh - 2\langle x, h\rangle Jx, \qquad F_{x\lambda}(x,\lambda)[h] = -h$$

for $x, h, \lambda \in \mathbb{R}$; moreover, for $\lambda_0 \in \mathbb{R}$,

$$F_x(0,\lambda)[h] = (A - \lambda I)h, \qquad F_{x\lambda}(0,\lambda)[h] = -h.$$

(i) Let λ_0 be not an eigenvalue of A.

Assertion: There are no nontrivial solutions of problem (3) *in a neighborhood of* $(0,\lambda_0)$.

This is a consequence of Corollary III.10 with $X := \mathbb{R}^n$, $L := A \in \mathcal{L}(X,X)$ and $h(x,\lambda) := |x|^2 Jx$; since by assumption $\lambda_0 \notin \sigma(A)$, the corollary states that no bifurcation occurs. □

(ii) Let λ_0 be a simple eigenvalue of A.

Assertion: Nontrivial solutions of problem (3) *form a continuous branch bifurcating from* $(0,\lambda_0)$.

We check the assumptions of the Crandall-Rabinowitz Bifurcation Theorem. Since A is assumed symmetric, there exist eigenpairs $(\psi_j, \mu_j) \in \mathbb{R}^n \times \mathbb{R}$ with $A\psi_j = \mu_j \psi_j$ and $\{\psi_1, ..., \psi_n\}$ being a complete orthonormal subset of $\mathbb{R}^n$.

Without loss of generality, $\mu_1 = \lambda_0$ and by assumption, we have $\mu_j \neq \lambda_0$ (simple eigenvalue). Thus,

$$\ker F_x(0,\lambda_0) = \ker(A - \lambda_0 I) = \text{span}\,\{\psi_1\},$$
$$\text{ran}\, F_x(0,\lambda_0) = \text{ran}(A - \lambda_0 I) = \text{span}\,\{\psi_2, ..., \psi_n\}$$

and we infer $\dim\ker F_x(0,\lambda_0) = \text{codim ran } F_x(0,\lambda_0) = 1$, so (S) holds. Moreover,

$$F_{x\lambda}(0,\lambda)[\psi_1] = -\psi_1 \notin \text{ran}\, F_x(0,\lambda_0),$$

and (T) is satisfied. Theorem IV.2 ensures the existence of a unique continuous branch of nontrivial solutions of problem (3) bifurcating from $(0,\lambda_0)$.

Alternatively: This is a direct consequence of Example IV.5 with $A(\lambda) := A - \lambda I$ since $\psi_1^T A'(\lambda_0)\psi_1 = -1 \neq 0$. □

Problem 18 (The Crandall-Rabinowitz Theorem for an ODE)

Let $g \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), (x, z, \lambda) \mapsto g(x, z, \lambda)$ be 2π-periodic in x with

$$g(x,0,\lambda) = 0, \quad g_z(x,0,\lambda) = 0, \quad g_{z\lambda}(x,0,\lambda) = 0 \quad \text{for all } x \in \mathbb{R}, \lambda \in \mathbb{R}.$$

In order to find nontrivial 2π-periodic solutions $u \in C^2(\mathbb{R})$ of the ODE

$$-u'' = \lambda u + g(x, u, \lambda) \qquad \text{on } \mathbb{R} \tag{4}$$

in a neighborhood of $(u_0, \lambda_0) = (0, 0)$, proceed as follows:

(a) Let $F : C^2_{\text{per}}(\mathbb{R}) \times \mathbb{R} \to C_{\text{per}}(\mathbb{R}),\ F(u, \lambda) := u'' + \lambda u + g(\,\cdot\,, u, \lambda)$ where

$$C^k_{\text{per}}(\mathbb{R}) := \left\{ u \in C^k(\mathbb{R}) :\ u(x) = u(x + 2\pi) \text{ for all } x \in \mathbb{R} \right\} \qquad \text{for } k \in \mathbb{N}_0.$$

Show that F is (partially) continuously Fréchet differentiable with respect to u.

(b) Show that $\ker(F_u(0, 0)) = \operatorname{span}\{\mathbb{1}\}$; $\operatorname{ran}(F_u(0, 0)) = \left\{ z \in C_{\text{per}}(\mathbb{R}) :\ \int_0^{2\pi} z(t)\,\mathrm{d}t = 0 \right\}$.

(c) Prove that there exist $\delta > 0$ and a continuous branch $(-\delta, \delta) \to C^2_{\text{per}}(\mathbb{R}) \times \mathbb{R},\ s \mapsto (\hat{u}(s), \hat{\lambda}(s))$ with the property that

$$\left\{ (\hat{u}(s), \hat{\lambda}(s)) :\ 0 < |s| < \delta \right\}$$

collects all nontrivial 2π-periodic solutions of problem (4) in a neighborhood of $(0, 0)$.

Solution

(a) We state (without proof) that, for $k \in \mathbb{N}_0$, the space $C^k_{\text{per}}(\mathbb{R})$ can be identified with a closed subspace of $\left(C^k([0, 2\pi]), \|\cdot\|_{C^k([0,2\pi])} \right)$ and hence is a Banach space. We denote as usual $C^0_{\text{per}}(\mathbb{R}) =: C_{\text{per}}(\mathbb{R})$ and $\|\cdot\|_{C^0([0,2\pi])} =: \|\cdot\|_\infty$.

The proof of continuous differentiability resembles that of Problem 13. We claim that

$$F_u(u, \lambda)[h] = h'' + \lambda h + g_z(\cdot, u, \lambda)h \quad (u, h \in C^2_{\text{per}}(\mathbb{R}); \lambda \in \mathbb{R}). \qquad (\heartsuit)$$

Then $F_u : C^2_{\text{per}}(\mathbb{R}) \to C_{\text{per}}(\mathbb{R})$ is linear and continuous. To justify $(\heartsuit)$, we estimate for $u, h \in C^2_{\text{per}}(\mathbb{R})$ and $\lambda \in \mathbb{R}$ with $\|h\|_{C^2([0,2\pi])} \leq 1$

$$\begin{aligned}
&\|F(u+h, \lambda) - F(u, \lambda) - [h'' + \lambda h + g_z(\cdot, u, \lambda)h]\|_\infty \\
&\quad = \|g(\cdot, u+h, \lambda) - g(\cdot, u, \lambda) + g_z(\cdot, u, \lambda)h\|_\infty \\
&\quad = \sup_{0 \leq x \leq 2\pi} \left| \int_0^1 \left[g_z(x, u(x) + \tau h(x), \lambda) - g_z(x, u(x), \lambda) \right] h(x)\,\mathrm{d}\tau \right| \\
&\quad = \sup_{0 \leq x \leq 2\pi} \left| \int_0^1 \int_0^\tau g_{zz}(x, u(x) + \sigma h(x), \lambda) h(x)^2\,\mathrm{d}\sigma \mathrm{d}\tau \right| \qquad \leq \frac{1}{2} C(u)\,\|h\|_\infty^2 \\
&\qquad \leq \frac{1}{2} C(u)\,\|h\|^2_{C^2([0,2\pi])}
\end{aligned}$$

where $C(u) := \sup\{|g_{zz}(x,z,\lambda)| : \ 0 \le x \le 2\pi, |z| \le \|u\|_\infty + 1\} < \infty$ due to continuity of g_{zz}. This proves Fréchet differentiability of F with respect to u.

To prove continuity of the derivative, we let $u, u_n = u + v_n \in C^2_{\text{per}}(\mathbb{R})$ with $u_n \to u$ in $C^2_{\text{per}}(\mathbb{R})$ and w.l.o.g. $\|v_n\|_\infty \le 1$, $\lambda \in \mathbb{R}$ and estimate similarly

$$\begin{aligned}
&\|F_u(u_n,\lambda) - F_u(u,\lambda)\|_{\mathcal{L}(C^2_{\text{per}}(\mathbb{R}),C_{\text{per}}(\mathbb{R}))} \\
&\quad = \sup_{h\in C^2_{\text{per}}(\mathbb{R}),\, \|h\|_{C^2([0,2\pi])}=1} \|F_u(u_n,\lambda)[h] - F_u(u,\lambda)[h]\|_\infty \\
&\quad = \sup_{h\in C^2_{\text{per}}(\mathbb{R}),\, \|h\|_{C^2([0,2\pi])}=1} \ \sup_{0\le x\le 2\pi} \big|\big(g_z(x,u(x)+v_n(x),\lambda) - g_z(x,u(x),\lambda)\big)h(x)\big| \\
&\quad \le \sup_{0\le x\le 2\pi} |g_z(x,u(x)+v_n(x),\lambda) - g_z(x,u(x),\lambda)| \\
&\quad = \sup_{0\le x\le 2\pi} \left| \int_0^1 g_{zz}(x,u(x)+\tau v_n(x),\lambda)\, v_n(x)\,\mathrm{d}\tau \right| \le C(u)\,\|v_n\|_\infty \le C(u)\,\|v_n\|_{C^2([0,2\pi])}\,.
\end{aligned}$$

Hence, $\|F_u(u_n,\lambda) - F_u(u,\lambda)\|_{\mathcal{L}(C^2_{\text{per}}(\mathbb{R}),C_{\text{per}}(\mathbb{R}))} \to 0$ as $n \to \infty$, which closes the proof.

(b) *(i) We show that* $\ker(F_u(0,0)) = \operatorname{span}\{\mathbb{1}\}$.

For $w \in C^2_{\text{per}}(\mathbb{R})$, the following are equivalent:

$$\begin{aligned}
w \in \ker F_u(0,0) \quad &\Leftrightarrow \quad w'' = 0 \quad \Leftrightarrow \quad \exists\, \alpha,\beta \in \mathbb{R}: \ w(x) = \alpha x + \beta \ \ (x \in \mathbb{R}) \\
\Leftrightarrow \quad &\exists\, \beta \in \mathbb{R}: \ w(x) = \beta \ \ (x \in \mathbb{R}) \quad \Leftrightarrow \quad w \in \operatorname{span}\{\mathbb{1}\}
\end{aligned}$$

where, in passing from the first to the second line, we exploited that w is periodic.

(ii) We show that $\operatorname{ran}(F_u(0,0)) = \left\{ z \in C_{\text{per}}(\mathbb{R}) : \ \int_0^{2\pi} z(x)\,\mathrm{d}x = 0 \right\}$.

Assume $z \in \operatorname{ran}(F_u(0,0))$. Then, there exists $w \in C^2_{\text{per}}(\mathbb{R})$ with $z = F_u(0,0)[w] = w''$ on $\mathbb{R}$. This yields, however,

$$\int_0^{2\pi} z(x)\,\mathrm{d}x = \int_0^{2\pi} w''(x)\,\mathrm{d}x = w'(2\pi) - w'(0) = 0$$

since w is periodic.

Conversely, assume that $z \in C_{\text{per}}(\mathbb{R})$ with $\int_0^{2\pi} z(x)\,\mathrm{d}x = 0$. We try to find $w \in C^2_{\text{per}}(\mathbb{R})$ with $z = F_u(0,0)[w] = w''$ on $\mathbb{R}$. Integration suggests the ansatz

$$\begin{aligned}
w(x) &= \alpha + \beta x + \int_0^x \int_0^y z(t)\,\mathrm{d}t\mathrm{d}y, \qquad \text{hence} \\
w'(x) &= \beta + \int_0^x z(t)\,\mathrm{d}t
\end{aligned}$$

with constants $\alpha = w(0)$ and $\beta = w'(0)$. By assumption on z, we have for $x \in \mathbb{R}$

$$w'(x+2\pi) - w'(x) = \int_0^{2\pi} z(x)\,\mathrm{d}x = 0,$$

i.e. w' has the asserted periodicity. Moreover, choosing $\beta := -\frac{1}{2\pi}\int_0^{2\pi}\int_0^y z(t)\,\mathrm{d}t\mathrm{d}y$, we also have, using that w' is 2π-periodic,

$$\begin{aligned} w(x+2\pi) - w(x) &= \int_x^{x+2\pi} w'(t)\,\mathrm{d}t = \int_0^{2\pi} w'(t)\,\mathrm{d}t \\ &= w(2\pi) - w(0) = 2\pi\beta + \int_0^{2\pi}\int_0^y z(t)\,\mathrm{d}t\mathrm{d}y \qquad = 0 \end{aligned}$$

and hence $w \in C^2_{\mathrm{per}}(\mathbb{R})$ with $z = F_u(0,0)[w]$.

(c) Finally, we intend to apply the Crandall-Rabinowitz Bifurcation Theorem. Looking back to the results of part (b), we prove that

$$\ker\left(F_u(0,0)\right) \oplus \operatorname{ran}\left(F_u(0,0)\right) = C_{\mathrm{per}}(\mathbb{R}).$$

Proof: If $w \in \ker\left(F_u(0,0)\right) \cap \operatorname{ran}\left(F_u(0,0)\right)$, we can find $\beta \in \mathbb{R}$ with $w \equiv \beta$; however, $0 = \int_0^{2\pi} w(x)\,\mathrm{d}x = 2\pi\beta$ proves that $w \equiv 0$ and hence the sum is direct.

For arbitrary $w \in C_{\mathrm{per}}(\mathbb{R})$, we split

$$w = w_0 + w_1 \qquad \text{where} \quad w_0(w) = \frac{1}{2\pi}\int_0^{2\pi} w(t)\,\mathrm{d}t, \quad w_1(x) = w(x) - \frac{1}{2\pi}\int_0^{2\pi} w(t)\,\mathrm{d}t.$$

Then $w_0 \in \ker\left(F_u(0,0)\right)$ and $w_1 \in \operatorname{ran}\left(F_u(0,0)\right)$. ■

We infer that $\dim\ker(F_u(0,0)) = \operatorname{codim}\operatorname{ran}(F_u(0,0)) = 1$, and the simplicity condition (S) holds. Moreover, transversality (T) is also satisfied since

$$F_{u\lambda}(0,0)[\mathbb{1}] = \mathbb{1} + g_{z\lambda}(\cdot,0,0)\mathbb{1} = \mathbb{1} \notin \operatorname{ran}\left(F_u(0,0)\right).$$

An application of Theorem IV.2 then closes the proof.

□