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Bifurcation Theory

Problem Sheet 8

Problem 22 (Eigenpairs of the Laplacian on a rectangular domain)

Let $a, b > 0$. Compute all eigenvalues and eigenfunctions of the Laplacian with homogeneous Dirichlet boundary conditions on the rectangular domain $\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2$, i.e. find $\lambda \in \mathbb{C}$, $u \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Proceed as follows:

(a) Consider the one dimensional boundary value problem

$$\begin{cases} -u'' = \lambda u & \text{in } (0,a), \\ u(0) = u(a) = 0 \end{cases}$$

and compute all eigenvalues and all eigenfunctions.

(b) Compute all eigenpairs (λ, u) of (1) where u is of the form $u(x,y) = v(x)w(y)$.

(c) Show that there are no other eigenpairs.

Hint: You may use without proof that $\left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{k\pi}{a} x\right) \right\}_{k \in \mathbb{N}}$ *is an orthonormal basis of* $L^2((0,a))$.

Problem 23 (Example IV.6 revisited)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $h \in L^\infty(\Omega)$ and consider the boundary value problem

$$\begin{cases} -\Delta u + \lambda u = h(x)\, |u|^{p-1}\, u & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{2}$$

Show that for all $p > 2$ there exists a curve of nontrivial solutions $(\widehat{u}(s), \widehat{\lambda}(s))$ of (2) bifurcating from any given simple eigenvalue of $(-\Delta)^{-1} : H_0^1(\Omega) \to H_0^1(\Omega)$.

Hint: Consider

$$-\Delta u + \lambda u = h(x)\chi\left(|u|^{p-1}\, u\right)$$

where $\chi \in C^\infty(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ *such that* $\chi(z) = z$ *for* $|z| \leq 1$.