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Bifurcation Theory

Solutions to Problem Sheet 9

Problem 24 (Necessary and sufficient conditions)

(a) Let $A \in \mathbb{R}^{n\times n}$ be symmetric and $J \in \mathbb{R}^{n\times n}$. Consider

$$F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad (x,\lambda) \mapsto Ax - \lambda x + |x|^2 Jx.$$

A necessary condition for bifurcation for $F(x,\lambda) = 0$ at $(0,\lambda_0)$ is that λ_0 is an eigenvalue of A. Show that this condition need not be sufficient.

(b) Let $F(x,\lambda) = x(g(\lambda) + x^2)$ where $g \in C^2(\mathbb{R})$ and $g(\lambda_0) = 0$ and consider the problem

$$(\star) \qquad F(x,\lambda) = 0.$$

(i) Under which conditions on g is $(0,\lambda_0)$ a bifurcation point for $(\star)$?

(ii) Under which conditions is the transversality condition (T) satisfied for $(\star)$?

(iii) Deduce that (T) is sufficient but in general not necessary for bifurcation.

Solution

(a) In problem 17, we have already seen, that a necessary condition for bifurcation at $(0,\lambda_0)$ is that λ_0 is an eigenvalue of A. Here, we will show, that this condition need not be sufficient, i.e. that for bifurcation from λ_0, we need that λ_0 is a simple eigenvalue of A.

Let $n = 2$ and consider

$$\begin{aligned} -x_2(x_1^2 + x_2^2) - \lambda x_1 &= 0, \\ x_1(x_1^2 + x_2^2) - \lambda x_2 &= 0, \end{aligned}$$

or equivalently

$$-\lambda x + |x|^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x = 0.$$

Here $A = 0$ and $\lambda_0 = 0$ is an eigenvalue of multiplicity 2. However, $(0, \lambda_0) \in \mathbb{R}^2 \times \mathbb{R}$ is not a bifurcation point. Let $x \in \mathbb{R}^2$ be a solution of the problem. Multiplying the first equation by $-x_2$ and the second equation by x_1 and adding them up leads to

$$\begin{aligned} 0 &= x_2^2(x_1^2 + x_2^2) + \lambda x_1 x_2 + x_1^2(x_1^2 + x_2^2) - \lambda x_1 x_2 \\ &= (x_1^2 + x_2^2)^2 \end{aligned}$$

and hence $x_1 = x_2 = 0$, meaning that $(0, \lambda_0) \in \mathbb{R}^2 \times \mathbb{R}$ is no bifurcation point.

(b) (i) Consider $F(x, \lambda) = x(g(\lambda) + x^2)$. Then we note that $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. We observe immediately that bifurcation form $(0, \lambda_0)$ occurs iff $g(\lambda) + x^2 = 0$ admits nontrivial solutions. This happens if there exists a sequence t_n with $t_n \to \lambda_0$ and $g(t_n) \nearrow 0$ for $n \to \infty$. Then we have a parametrization of nontrivial solutions of the form $\{(\pm\sqrt{-g(t_n)}, t_n) : t_n \in \mathbb{R}\}$.

(ii) F is twice continuously differentiable with

$$F_x(x, \lambda) = g(\lambda) + 3x^2,$$

and

$$F_{x\lambda}(x, \lambda) = g'(\lambda)$$

for $x, \lambda \in \mathbb{R}$, and in particular for $\lambda_0 \in \mathbb{R}$,

$$F_x(0, \lambda_0) = 0$$

and

$$F_{x\lambda}(0, \lambda_0) = g'(\lambda_0).$$

Hence, we have $\ker(F_x(0, \lambda_0)) = \mathbb{R}$. Thus the transversality condition is fulfilled, iff $g'(\lambda_0) \neq 0$ as in this case

$$F_{x\lambda}(0, \lambda_0) = g'(\lambda_0) \notin \{0\} = \operatorname{ran}(F_x(0, \lambda_0)).$$

(iii) However, without the transversality condition, we don't know whether $(0, \lambda_0)$ is a bifurcation point or not. Consider for example $g(\lambda) = \lambda^2$, then we observe that $(0, 0)$ is not a bifurcation point as $F(x, \lambda) = 0$ implies $x = 0$ as $\lambda^2 + x^2 \geq 0$. In contrast, consider $g(\lambda) = \lambda^3$. In this case $(0, 0)$ is a bifurcation point, as there exists a nontrivial curve of solutions parametrized by $\{(t^3, -t^2) : t \in \mathbb{R}\}$.

$\square$

Problem 25 (Bifurcation from ∞ in finite dimensions)

Consider the nonlinear system

$$\begin{cases} (1-\lambda)x_1 + \frac{x_2}{x_1^2+x_2^2} = 0, \\ (1-2\lambda)x_2 + \frac{x_1}{x_1^2+x_2^2} = 0. \end{cases}$$

Show that bifurcation from ∞ occurs for $\lambda_0 = 1$ and $\lambda_0 = \frac{1}{2}$.

Solution

Here, we will only consider the case $\lambda_0 = 1$. The case $\lambda_0 = \frac{1}{2}$ can be treated analogously.

Solving the problem is equivalent to finding zeros of the function

$$F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \quad F(x_1, x_2, \lambda) := \begin{pmatrix} (1-\lambda)x_1 + \frac{x_2}{x_1^2+x_2^2} \\ (1-2\lambda)x_2 + \frac{x_1}{x_1^2+x_2^2} \end{pmatrix}$$

or equivalently

$$F(x_1, x_2, \lambda) = L(\lambda)x + R(x, \lambda)$$

where

$$L(\lambda) = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-2\lambda \end{pmatrix}, \quad R(x, \lambda) = \begin{pmatrix} \frac{x_2}{x_1^2+x_2^2} \\ \frac{x_1}{x_1^2+x_2^2}. \end{pmatrix}$$

Then

$$L(\lambda_0) = \begin{pmatrix} 0 & 0 \\ 0 & 1-2\lambda_0 \end{pmatrix}$$

and hence

$$\ker(L(\lambda_0)) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

and

$$\operatorname{ran}(L(\lambda_0)) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Moreover with $\varphi := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have

$$L'(\lambda_0)\varphi = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \notin \operatorname{ran}(L(\lambda_0)).$$

Hence conditions (i) and (ii) of Theorem IV.10 are fulfilled.

Now we check condition (iii). We have

$$\left|sR\left(\frac{x}{s},\lambda\right)\right| = s^2\sqrt{\frac{x_1^2+x_2^2}{(x_1^2+x_2^2)^2}} = s^2\frac{1}{|x|} \leq 2s^2 \qquad \to 0$$

for $s \to 0$ and $\frac{1}{2} \leq |x| \leq 2$ in a neighborhood of φ.

(Note that as $|\varphi| = 1$ we can assume $\frac{1}{2} \leq |x| \leq 2$ in a neighborhood of φ).

We also have

$$\left|sR_\lambda\left(\frac{x}{s},\lambda\right)\right| = 0$$

and finally together with

$$R_x\left(\frac{x}{s},\lambda\right) = s^2\frac{1}{(x_1^2+x_2^2)^2}\begin{pmatrix}-2x_1x_2 & x_1^2-x_2^2\\ x_2^2-x_1^2 & -2x_1x_2\end{pmatrix}$$

we get

$$\begin{aligned}\left|R_x\left(\frac{x}{s},\lambda\right)\psi\right| &= s^2\frac{1}{(x_1^2+x_2^2)^2}\sqrt{4x_1^2x_2^2\psi_1^2+(x_1^2-x_2^2)^2\psi_2^2+(x_2^2-x_1^2)^2\psi_1^2+4x_1^2x_2^2\psi_2^2}\\ &= s^2\frac{1}{(x_1^2+x_2^2)^2}\sqrt{4x_1^2x_2^2+(x_1^2-x_2^2)^2}\sqrt{\psi_1^2+\psi_2^2}\\ &= s^2\frac{1}{(x_1^2+x_2^2)^2}\sqrt{(x_1^2+x_2^2)^2}\,|\psi|\\ &= s^2\frac{1}{(x_1^2+x_2^2)}\,|\psi|\end{aligned}$$

which yields

$$\left\|R_x\left(\frac{x}{s},\lambda\right)\right\| = s^2\frac{1}{(x_1^2+x_2^2)} \leq 4s^2 \qquad \to 0$$

for $s \to 0$ and $\frac{1}{2} \leq |x| \leq 2$ in a neighborhood of $\varphi = \begin{pmatrix}1\\0\end{pmatrix}$. This finally proves (iii) of Theorem IV.10 and closes the proof.

□

Problem 26 (Example IV.11 revisited)

Consider the same example as in the lecture but this time with homogeneous Dirichlet boundary conditions on $\Omega = (0,1)$

$$\begin{cases} -u'' = \lambda u + \frac{a(x,\lambda)u}{1+u^2} & \text{in } \Omega, \\ u(0) = u(1) = 0 \end{cases}$$

for $a \in C^1(\overline{\Omega} \times \mathbb{R})$. Assume that $a(x,\lambda) = 0$ whenever x is near 0 or near 1. Use Theorem IV.10 to show that bifurcation from infinity occurs at $\lambda_0 = \pi^2$.

Solution

Define $X := C^2_{\text{Dirichlet}}(\overline{\Omega})$ and $Z := C(\overline{\Omega})$. Solving the problem is equivalent to finding zeros of the function

$$F : X \times \mathbb{R} \to Z, \quad F(u,\lambda) := u'' + \lambda u + \frac{a(x,\lambda)u}{1+u^2}$$

or equivalently

$$F(u,\lambda) = L(\lambda)u + R(u,\lambda)$$

where

$$L(\lambda) : u \mapsto u'' + \lambda u, \quad R(u,\lambda) = \frac{a(\cdot,\lambda)u}{1+u^2}.$$

Assertion: $\ker(L(\lambda_0)) = \operatorname{span}\{\phi\}$ *where* $\phi = \sin(\pi x)$, $\operatorname{ran}(L(\lambda_0)) = \{f \in C([0,1]) | f \perp \ker(L(\lambda_0))\}$, *i.e.* $f \in \operatorname{ran}(L(\lambda_0))$ *iff* $\int_0^1 f(x)\sin(\pi s)ds = 0$ *and* $\ker(L(\lambda_0)) \oplus \operatorname{ran}(L(\lambda_0)) = C([0,1])$

From problem 22 we know that $\ker(L(\lambda_0)) = \operatorname{span}\{\phi\}$ where $\phi = \sin(\pi x)$.

Now we will show that $\operatorname{ran}(L(\lambda_0)) = \{f \in C([0,1]) | f \perp \ker(L(\lambda_0))\}$, i.e. $f \in \operatorname{ran}(L(\lambda_0))$ iff $\int_0^1 f(x)\sin(\pi s)ds = 0$. To this end we proceed as in problem 18.

First, we assume $z \in \operatorname{ran}(L(\lambda_0))$. Then there exists $w \in X$ with $L(\lambda_0)w = z$. Hence,

$$w \in C^2([0,1]), \quad \begin{cases} w'' + \lambda_0 w = z & \text{in } (0,1). \\ w(0) = w(1) = 0. \end{cases}$$

We calculate, using integration by parts,

$$
\begin{aligned}
&\int_0^1 w''(x)\sin(\pi x) + \pi^2 w(x)\sin(\pi x)\,dx \\
=&\left[w'(x)\sin(\pi x)\right]_0^1 - \pi\int_0^1 w'(x)\cos(\pi x)\,dx + \pi^2\int_0^1 w(x)\sin(\pi x)\,dx \\
=&0 - \pi^2\int_0^1 w(x)\sin(\pi x)\,dx + \pi^2\int_0^1 w(x)\sin(\pi x)\,dx = 0.
\end{aligned}
$$

On the other hand, we consider $z \in Z$ with $\int_0^1 z(x)\cos(\pi x)dx = 0$. For arbitrary $\alpha \in \mathbb{R}$, the Picard-Lindelöf Theorem guarantees existence of a unique $w_\alpha \in C^2([0, 2\pi])$ satisfying the initial value problem

$$
\begin{cases} w_\alpha'' + \lambda_0 w_\alpha = z & \text{in } (0,1), \\ w_\alpha(0) = 0, w_\alpha'(0) = \alpha. \end{cases}
$$

We will show that the condition on z then yields $w_\alpha(1) = 0$. To this end, we consider

$$
\begin{aligned}
w_\alpha(1) &= -w_\alpha(1)\cos\pi + w_\alpha(0)\cos(0) \\
&= -\int_0^1 \frac{d}{dx}\left[w_\alpha(x)\cos(\pi x)\right]\,dx \\
&= -\int_0^1 \left[w_\alpha'(x)\cos(\pi x) - \pi w_\alpha(x)\sin(\pi x)\right]\,dx \\
&= \frac{1}{\pi}\int_0^1 \left[w_\alpha''(x)\sin(\pi x) + \pi^2 w_\alpha(x)\sin(\pi x)\right]\,dx \\
&= \frac{1}{\pi}\int_0^1 z(x)\sin(\pi x)\,dx = 0,
\end{aligned}
$$

where we used integration by parts and the differential equation solved by w_α in the end.

The assertion on the direct sum is done as in problem 21.

Assertion: Bifurcation from infinity occurs at $\lambda_0 = \pi^2$

We have seen that $\dim\ker(L(\lambda_0)) = \operatorname{codim}\operatorname{ran}(L(\lambda_0)) = 1$ and

$$
L'(\lambda_0)[\phi] \notin \operatorname{ran}(L(\lambda_0)).
$$

Thus, conditions (i) and (ii) of Theorem IV.10 hold.

Now we will prove condition (iii). To do so, define

$$
\varepsilon := \inf_{\lambda\in\mathbb{R}} \operatorname{dist}(\operatorname{supp}(a(\cdot,\lambda), \partial\Omega)) > 0, \quad M := \|a\|_{L^\infty([0,1]\times\mathbb{R})} + \|a_\lambda\|_{L^\infty([0,1]\times\mathbb{R})}.
$$

Then we have $\phi(x) \geq 2\sin(\varepsilon) =: 2\delta > 0$ for $x \in [\varepsilon, 1-\varepsilon]$ so that all $(u,\lambda) \in U := \{(u,\lambda) \in X \times \mathbb{R} : \|u - \phi\|_X \leq \delta\}$ satisfy

$$
0 < \delta \leq u(x) \leq 1 + \delta < 2 \qquad (x \in [\varepsilon, 1-\varepsilon]).
$$

Since $a(x,\lambda) = 0$ for $x \in [0,1] \setminus [\varepsilon, 1-\varepsilon]$ we have

$$sR\left(\frac{u}{s},\lambda\right)(x) = sR_\lambda\left(\frac{u}{s},\lambda\right)(x) = R_x\left(\frac{u}{s},\lambda\right)[\psi](x) = 0 \qquad (x \in [0,1] \setminus [\varepsilon, 1-\varepsilon])$$

for all such x and all $\psi \in X$. Therefore it remains to estimate the nonlinear term on the interval $[\varepsilon, 1-\varepsilon]$. We get for all $(u,\lambda) \in U$

$$\begin{aligned}
\left\|sR\left(\frac{u}{s},\lambda\right)\right\|_Z &\le M\,\|u\|_{L^\infty([\varepsilon,1-\varepsilon])}\left\|\frac{s^2}{s^2+u^2}\right\|_{L^\infty([\varepsilon,1-\varepsilon])} \le \frac{2M}{\delta^2}s^2 \to 0,\\
\left\|sR_\lambda\left(\frac{u}{s},\lambda\right)\right\|_Z &\le M\,\|u\|_{L^\infty([\varepsilon,1-\varepsilon])}\left\|\frac{s^2}{s^2+u^2}\right\|_{L^\infty([\varepsilon,1-\varepsilon])} \le \frac{2M}{\delta^2}s^2 \to 0,\\
\left\|R_u\left(\frac{u}{s},\lambda\right)\right\|_{\mathcal{L}(X,Z)} &= \sup_{\|\psi\|_X=1}\left\|a(\cdot,\lambda)\frac{s^4-u^2s^2}{(s^2+u^2)^2}\psi\right\|_Z\\
&\le M\cdot\sup_{\|\psi\|_X=1}\|\psi\|_{L^\infty([\varepsilon,1-\varepsilon])}\left\|\frac{s^4-u^2s^2}{(s^2+u^2)^2}\right\|_{L^\infty([\varepsilon,1-\varepsilon])}\\
&\le M\left\|\frac{s^2}{s^2+u^2}\right\|_{L^\infty([\varepsilon,1-\varepsilon])}\\
&\le \frac{M}{\delta^2}s^2 \to 0.
\end{aligned}$$

Hence, we conclude that condition (iii) of Theorem IV.10 can also be applied which closes the proof. □

Remark: The condition on a can be omitted:

Let $\alpha > 0$ such that

$$\phi(x) = \sin(\pi x) \ge \alpha x(1-x).$$

Proposition: Let X be as above.

(i) For all $\psi \in X, \|\psi\|_X \le 1$ we habe $|\psi(x)| \le 2x(1-x)$.

(ii) $\|u-\phi\|_X \le \frac{\alpha}{4}$ implies $u(x) \ge \frac{\alpha}{4}x(1-x)$.

Proof

(ii) For u as above and $0 \le x \le \frac{1}{2}$ we have

$$\begin{aligned}
u(x) &= \int_0^x u'(t)\,dt \ge \int_0^x \phi'(t) - \frac{\alpha}{4}\,dt\\
&\ge \phi(x) - \frac{\alpha}{4}x \ge \alpha x(1-x)\cdot(1-\frac{1}{4(1-x)})\\
&\ge \frac{\alpha}{4}x(1-x)
\end{aligned}$$

(i) As above

$$|\psi(x)| \le \max\{x, 1-x\} \le 2x(1-x)$$

$\square$

Let $U := \{(u, \lambda) \in X \times \mathbb{R} : \|u - \phi\| \leq \frac{\alpha}{4}\}$ and $\varepsilon > 0$. Choose $s_\varepsilon := \frac{\alpha\varepsilon}{8(1+\|a\|_\infty)^{3/2}}$. Then we have for all $\psi \in X$ with $\|\psi\|_X \leq 1$ and $u \in X$ with $\|u - \phi\|_X \leq \frac{\alpha}{4}$:

- If $2x(1-x) \leq \frac{\varepsilon}{1+\|a\|_\infty}$

$$\begin{aligned}\Big|\frac{a(x,\lambda)(s^4 - u(x)^2 s^2)}{(s^2 + u(x)^2)^2}\psi(x)\Big| &\leq \|a\|_\infty \frac{s^2}{s^2 + u(x)^2}|\psi(x)| \\ &\leq \|a\|_\infty \cdot 1 \cdot 2x(1-x) \\ &\leq \varepsilon.\end{aligned}$$

- If $2x(1-x) \geq \frac{\varepsilon}{1+\|a\|_\infty}$, we get

$$u(x) \geq \frac{\alpha}{4}x(1-x) \geq \frac{\alpha\varepsilon}{8(1+\|a\|_\infty)} =: \delta$$

  and thus for $|s| \leq s_\varepsilon$

$$\begin{aligned}\Big|\frac{a(x,\lambda)(s^4 - u(x)^2 s^2)}{(s^2 + u(x)^2)^2}\psi(x)\Big| &\leq \|a\|_\infty \frac{s^2}{s^2 + u(x)^2}|\psi(x)| \\ &\leq \frac{\|a\|_\infty}{\delta^2} \cdot s^2 \cdot 1 \\ &\leq \varepsilon.\end{aligned}$$

These estimates prove that estimates on R requested in Theorem IV.10 work and thus it can be applied.

Note: It is not possible to prove bifurcation of sign-changing solutions with this technique.