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Bifurcation Theory

Solutions to Problem Sheet 10

We recall the **Crandall-Rabinowitz Theorem:** Let X, Z be Banach spaces and assume for some $\lambda_0 \in \mathbb{R}$

(i) $F \in C^2(X \times \mathbb{R}, Z)$ and (ii) $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$,

(S) $F_x(0, \lambda_0) \in \mathcal{L}(X, Z)$ is a 1-1-Fredholm operator,

(T) $F_{x\lambda}(0, \lambda_0)[\phi] \notin \operatorname{ran}(F_x(0, \lambda_0))$ where $\ker(F_x(0, \lambda_0)) = \operatorname{span}\{\phi\}$.

Then there is a curve $(\hat{x}, \hat{\lambda}) \in C^1((-\varepsilon, \varepsilon), X \times \mathbb{R})$ with $\hat{\lambda}(0) = \lambda_0$, $\hat{x}(0) = 0$, $\hat{x}'(0) = \phi$ and

$$F(\hat{x}(s), \hat{\lambda}(s)) = 0 \qquad \text{for all } s \in (-\varepsilon, \varepsilon).$$

Problem 27 (Crandall-Rabinowitz Theorem via Lyapunov-Schmidt reduction)

(a) Prove the Crandall-Rabinowitz Theorem using Lyapunov-Schmidt reduction. Proceed as follows:

(i) Decompose the spaces X and Z according to the Lyapunov-Schmidt reduction and write down the reduction equation $g(s, \lambda) = 0$, i.e. find a function $g : U \to \mathbb{R}$ where U is a neighborhood of $(0, \lambda_0)$ in $\mathbb{R} \times \mathbb{R}$ such that $F(x, \lambda) = 0$ admits nontrivial solutions (x, λ) bifurcating from $(0, \lambda_0)$ iff $g(s, \lambda) = 0$ admits nontrivial solutions (s, λ) bifurcating from $(0, \lambda_0)$.

(ii) Show that the transversality condition (T) is equivalent to the condition

$$g_{s\lambda}(0, \lambda_0) \neq 0.$$

(iii) Apply the Implicit Function Theorem to conclude the statement.

(b) Now show a slightly different version of the Crandall-Rabinowitz Theorem. Assume that $F \in C^4(X \times \mathbb{R}, Z)$ and that (ii) and (S) hold as well as

(T') $\quad F_{x\lambda}(0, \lambda_0)[\phi], F_{x\lambda\lambda}(0, \lambda_0)[\phi] \in \operatorname{ran}(F_x(0, \lambda_0)), \quad F_{x\lambda\lambda\lambda}(0, \lambda_0)[\phi] \notin \operatorname{ran}(F_x(0, \lambda_0))$.

Prove that in this case $(0, \lambda_0) \in X \times \mathbb{R}$ is a bifurcation point.

Hint: Prove that $g_{s\lambda}(0, \lambda_0) = g_{s\lambda\lambda}(0, \lambda_0) = 0$ *and* $g_{s\lambda\lambda\lambda}(0, \lambda_0) \neq 0$.

Solution

(a) (i) Let $L := F_x(0,\lambda_0) : X \to X$. Then we can decompose the spaces X and Z according to the Lyapunov-Schmidt reduction and get

$$X = \ker(L) \oplus \widetilde{X} =: X_1 \oplus X_2,$$
$$Z = \operatorname{span}\{\phi^\star\} \oplus \operatorname{ran}(L).$$

By assumptions,

$$\ker(F_x(0,\lambda_0)) = \{\phi\}$$

and by the Hahn-Banach Theorem there exists some $\phi' \in Z' \backslash \{0\}$ such that $\ker \phi' = \operatorname{ran}(F_x(0,\lambda_0))$. The reduction equation now reads as

$$g(s,\lambda) = \langle F(s\phi + \psi(s\phi,\lambda),\lambda), \phi' \rangle = 0$$

with trivial solution $s = 0$, and we now look for nontrivial solutions.

From the Implicit Function Theorem we know that there is a unique solution

$$\psi : V_1 \times V \to V_2$$

satisfying

$$PF(x_1 + \psi(x_1,\lambda),\lambda) = 0,$$

where V_i is a neighborhood of 0 in $U \cap X_i$, $i = 1, 2$, and V is a neighborhood of λ_0. Here $Pz = z - \langle z, \phi' \rangle \phi^\star$, where $\phi^\star \in Z$ is chosen such that $Z = \operatorname{ran}(L) \oplus \operatorname{span}\{\phi^\star\}$ and $\langle \phi^\star, \phi' \rangle = 1$.

(ii) *Assertion:* $\psi_{x_1}(0,\lambda_0) = 0$.

Since $F(0,\lambda) = 0$, we have $\psi(0,\lambda) = 0$. Using again the Implicit Function Theorem, we obtain

$$\psi'(0,\lambda_0)[\overline{x_1},\overline{\lambda}] = -(PF_x(0,\lambda_0))^{-1}[PF_x(0,\lambda_0)[\overline{x_1}] + PF_\lambda(0,\lambda_0)[\overline{\lambda}]]$$

for all $(\overline{x_1},\overline{\lambda}) \in X_1 \times \mathbb{R}$. As $F(0,\lambda) = 0$ for all λ, we have $F_\lambda(0,\lambda) = 0$ and thus

$$\psi'(0,\lambda_0) = 0,$$

provided $\overline{x_1} \in X_1 = \ker(F_x(0,\lambda_0))$.

Assertion: (T) is equivalent to the condition

$$g_{s\lambda}(0,\lambda_0) \neq 0.$$

We observe

$$\begin{aligned} g_{s\lambda}(0,\lambda_0) &= \langle F_{x\lambda}(0,\lambda_0)[\phi + \psi_{x_1}(0,\lambda_0)[\phi]] + F_x(0,\lambda_0)[\psi_{x_1\lambda}(0,\lambda_0)[\phi]], \phi' \rangle \\ &= \langle F_{x\lambda}(0,\lambda_0)[\phi], \phi' \rangle \\ &\neq 0 \end{aligned}$$

iff $F_{x\lambda}(0,\lambda_0)[\phi] \notin \operatorname{ran}(F_x(0,\lambda_0))$.

(iii) *Assertion: $h \in C^1(B_\eta)$ where*

$$h(s,\lambda) = \begin{cases} \frac{1}{s} g(s,\lambda) & s \neq 0, \\ g_s(0,\lambda) & s = 0, \end{cases}$$

and $B_\eta = \{(s,\lambda) \in \mathbb{R}^2 | \, |s|^2 + |\lambda - \lambda_0|^2 < \eta^2\}$ for η small enough.

We only need to prove that h is continuously differentiable in $(0,\lambda)$. By definition we have

$$\lim_{s \to 0} \frac{1}{s} g(s,\lambda) = g_s(0,\lambda)$$

and therefore h is continuous. Moreover,

$$\begin{aligned} h_s(0,\lambda) &= \lim_{s \to 0} \frac{1}{s} \left[h(s,\lambda) - h(0,\lambda) \right] \\ &= \lim_{s \to 0} \frac{1}{s^2} \left[g(s,\lambda) - g(0,\lambda) - g_s(0,\lambda) s \right] \\ &= \frac{1}{2} g_{ss}(0,\lambda), \end{aligned}$$

and thus

$$\begin{aligned} h_s(s,\lambda) - h_s(0,\lambda) &= \frac{1}{s^2} \left[g_s(s,\lambda) s - g(s,\lambda) - \frac{1}{2} g_{ss}(0,\lambda) s^2 \right] \\ &= o(1) \end{aligned}$$

as $|s| \to 0$. Since

$$h_\lambda(0,\lambda) = g_{s\lambda}(0,\lambda),$$

we have

$$h_\lambda(s,\lambda) - h_\lambda(0,\lambda) = \frac{1}{s} g_\lambda(s,\lambda) - g_{s\lambda}(0,\lambda) \to 0$$

as $s \to 0$ and consequently $h \in C^1(B_\eta)$.

Assertion: There is a C^1-curve $\hat{\lambda} = \hat{\lambda}(s)$, $|s| < \delta$, satisfying

$$\begin{cases} h(s,\hat{\lambda}(s)) = 0, \\ \hat{\lambda}(0) = \lambda_0. \end{cases}$$

We know from (ii) that

$$h_\lambda(0,\lambda_0) = g_{s\lambda}(0,\lambda_0) \neq 0.$$

Now we can apply the Implicit Function Theorem and obtain with $\hat{x}(s) := s\phi + \psi(s\phi, \widehat{\lambda}(s))$ where ψ is chosen as in part (i)

$$F(\hat{x}(s), \hat{\lambda}(s)) = 0,$$

which closes the proof.

(b) We state (without proof) that $h \in C^4(B_\eta)$ and as before we calculate

$$\begin{aligned} h_\lambda(0,\lambda_0) =& g_{s\lambda}(0,\lambda_0) \\ =& \langle F_{x\lambda}(0,\lambda_0)[\phi + \psi_{x_1}(0,\lambda_0)[\phi]] + F_x(0,\lambda_0)[\psi_{x_1\lambda}(0,\lambda_0)[\phi]], \phi'\rangle = 0, \end{aligned}$$

and

$$\begin{aligned} h_{\lambda\lambda}(0,\lambda_0) =& g_{s\lambda\lambda}(0,\lambda_0) \\ =& \langle F_{x\lambda\lambda}(0,\lambda_0)[\phi + \psi_{x_1}(0,\lambda_0)[\phi]] + 2F_{x\lambda}(0,\lambda_0)[\phi + \psi_{x_1\lambda}(0,\lambda_0)[\phi]] \\ &+ F_x(0,\lambda_0)[\psi_{x_1\lambda\lambda}(0,\lambda_0)[\phi]], \phi'\rangle = 0, \end{aligned}$$

as well as

$$\begin{aligned} h_{\lambda\lambda\lambda}(0,\lambda_0) =& g_{s\lambda\lambda\lambda}(0,\lambda_0) \\ =& \langle F_{x\lambda\lambda\lambda}(0,\lambda_0)[\phi + \psi_{x_1}(0,\lambda_0)[\phi]] + 3F_{x\lambda\lambda}(0,\lambda_0)[\phi + \psi_{x_1\lambda}(0,\lambda_0)[\phi]] \\ &+ 3F_{x\lambda}(0,\lambda_0)[\phi + \psi_{x_1\lambda\lambda}(0,\lambda_0)[\phi]] + F_x(0,\lambda_0)[\psi_{x_1\lambda\lambda\lambda}(0,\lambda_0)[\phi]], \phi'\rangle \\ =& \langle F_{x\lambda\lambda\lambda}(0,\lambda_0)[\phi], \phi'\rangle \neq 0. \end{aligned}$$

Hence,

$$h(0,\lambda_0 - \varepsilon) \cdot h(0,\lambda_0 + \varepsilon) = g_s(0,\lambda_0 - \varepsilon) \cdot g_s(0,\lambda_0 + \varepsilon) < 0$$

for some $\varepsilon > 0$. This finally yields, that for some $\delta_\varepsilon > 0$ and all s with $|s| < \delta_\varepsilon$ we find a solution μ_s with

$$\begin{cases} h(s,\mu_s) = 0 \\ \mu_0 = \lambda_0 \end{cases}$$

in this interval. The rest of the proof is done analogously to part (a).

□

Remark: One can use Taylor's theorem to get

$$h(0,\lambda_0 - \varepsilon) \cdot h(0,\lambda_0 + \varepsilon) = g_s(0,\lambda_0 - \varepsilon) \cdot g_s(0,\lambda_0 + \varepsilon) < 0.$$

We have

$$h(0,\lambda) = h(0,\lambda_0) + h_{\lambda\lambda\lambda}(0,\xi(\lambda))\frac{(\lambda-\lambda_0)^3}{3!}$$

where ξ between λ and λ_0. As $h_{\lambda\lambda\lambda}$ is continuous in λ_0 we have $h_{\lambda\lambda\lambda}(0,\xi(\lambda)) \to h_{\lambda\lambda\lambda}(0,\lambda_0) \neq 0$ for $\lambda \to \lambda_0$. Additionally $h_{\lambda\lambda\lambda}$ does not change sign in a suitable neighborhood of λ_0. W.l.o.g we now assume $h_{\lambda\lambda\lambda}(0,\xi(\lambda+\varepsilon)) > 0$ for some $\varepsilon > 0$ and we obtain

$$h(0,\lambda_0+\varepsilon) = h(0,\lambda_0) + \frac{1}{3!}h_{\lambda\lambda\lambda}(0,\xi(\lambda_0+\varepsilon))\varepsilon^3 > h(0,\lambda_0) = 0$$

and

$$h(0,\lambda_0-\varepsilon) = h(0,\lambda_0) + \frac{1}{3!}h_{\lambda\lambda\lambda}(0,\xi(\lambda_0-\varepsilon))(-\varepsilon)^3 < h(0,\lambda_0) = 0.$$