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Bifurcation Theory

Problem Sheet 11

We recall **assumptions (H1)-(H4)**:

(H1) $f \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$.

(H2) (Simplicity:) There are nontrivial $\phi, \psi \in \mathbb{C}^n$ with $\langle \psi, \phi \rangle_{\mathbb{C}^n} = 1$ and $\beta > 0$ such that

$$\ker(f_x(0,\lambda_0) - \mathrm{i}\beta) = \operatorname{span}\{\phi\}, \quad \ker(f_x(0,\lambda_0)^T + \mathrm{i}\beta) = \operatorname{span}\{\psi\}.$$

(H3) (Nonresonance:) For all $k \in \mathbb{Z}$ such that $|k| \neq 1$ we have $\ker(f_x(0,\lambda_0) - \mathrm{i}k\beta) = \{0\}$.

(H4) (Transversality:) $\operatorname{Re}\left(\langle f_{x\lambda}(0,\lambda_0)\phi, \psi \rangle_{\mathbb{C}^n}\right) \neq 0$.

Problem 28 (Proof of Proposition V.1)

Let $\xi \in \mathbb{C}^n$. Then for ϕ, ψ as in (H2), we get

(i) $y_k(t) = \left(e^{\mathrm{i}kt} id - e^{\frac{t}{\beta}A}\right)(A - \mathrm{i}k\beta)^{-1}\xi$ if $|k| \neq 1$,

(ii) $y_1(t) = \left(e^{\mathrm{i}t} id - e^{\frac{t}{\beta}A}\right) w - \frac{t}{\beta} e^{\mathrm{i}t} \langle \psi, \xi \rangle_{\mathbb{C}^n} \phi$
where $w \in \mathbb{C}^n$ satisfies $(A - \mathrm{i}\beta)w = \xi - \langle \psi, \xi \rangle_{\mathbb{C}^n} \phi$.

Problem 29 (Simplicity (H2))

Let $\beta > 0$ and $A \in \mathbb{R}^{n \times n}$ with

$$\ker(A - \mathrm{i}\beta) = \operatorname{span}\{\phi\}, \quad \ker(A^T + \mathrm{i}\beta) = \operatorname{span}\{\psi\}, \quad \langle \psi, \phi \rangle_{\mathbb{C}^n} = 1.$$

Show that

$$\psi \cdot \phi = 0.$$

Problem 30 (Assumptions (H1)-(H4))

Consider the differential equation

$$-y'' = g(y', y, \lambda).$$

Under which conditions on g are the assumptions (H1)-(H4) valid for some given $\lambda_0 \in \mathbb{R}$, $\beta > 0$?

Hint: Write the differential equation as a two-dimensional dynamical system in the variable $x = (y, y')$.