Karlsruhe Institute of Technology (KIT)
Institute for Analysis
Dr. Rainer Mandel
M.Sc. Janina Gärtner, M.Sc. Dominic Scheider

Summer Term 2017
05.07.2017

Bifurcation Theory

Solutions to Problem Sheet 11

Problem 28 (Proof of Proposition V.1)

Let $\xi \in \mathbb{C}^n$. Then for ϕ, ψ as in (H2), we get

(i) $y_k(t) = \left(e^{\mathrm{i}kt} id - e^{\frac{t}{\beta}A}\right)(A - \mathrm{i}k\beta)^{-1}\xi$ if $|k| \neq 1$,

(ii) $y_1(t) = \left(e^{\mathrm{i}t} id - e^{\frac{t}{\beta}A}\right) w - \frac{t}{\beta} e^{\mathrm{i}t} \langle \psi, \xi \rangle_{\mathbb{C}^n} \phi$
where $w \in \mathbb{C}^n$ satisfies $(A - \mathrm{i}\beta)w = \xi - \langle \psi, \xi \rangle_{\mathbb{C}^n} \phi$.

Solution

As in the lecture let

$$y_k(t) := -\frac{1}{\beta}\int_0^t \exp\left(\frac{t-\tau}{\beta}A\right)\xi e^{\mathrm{i}k\tau}\,d\tau \quad (\xi \in \mathbb{C}^n).$$

(i) For $|k| \neq 1$ let $\eta := (A - \mathrm{i}k\beta)^{-1}\xi$. Then we have

$$\begin{aligned}
y_k(t) &= e^{\mathrm{i}kt}\int_0^t -\frac{1}{\beta}\exp\left(\frac{t-\tau}{\beta}A + \mathrm{i}k(\tau - t)\right)\xi\,d\tau \\
&= e^{\mathrm{i}kt}\int_0^t -\frac{1}{\beta}\exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}k\beta)\right)\xi\,d\tau \\
&= e^{\mathrm{i}kt}\int_0^t -\frac{1}{\beta}\exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}k\beta)\right)(A - \mathrm{i}k\beta)\eta\,d\tau \\
&= e^{\mathrm{i}kt}\left[\exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}k\beta)\right)\eta\right]_0^t \\
&= e^{\mathrm{i}kt}\left(\eta - \exp\left(\frac{t}{\beta}(A - \mathrm{i}k\beta)\right)\eta\right) \\
&= e^{\mathrm{i}kt}\eta - \exp\left(\frac{t}{\beta}A\right)\eta \\
&= \left(e^{\mathrm{i}kt} id - e^{\frac{t}{\beta}A}\right)(A - \mathrm{i}k\beta)^{-1}\xi.
\end{aligned}$$

(ii) For $k = 1$ we get similarly

$$\begin{aligned}
y_1(t) &= e^{\mathrm{i}t} \int_0^t -\frac{1}{\beta} \exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}\beta)\right) \left((A-\mathrm{i}\beta)w + \langle \psi, \xi \rangle_{\mathbb{C}^n} \phi\right) d\tau \\
&= e^{\mathrm{i}t} \int_0^t -\frac{1}{\beta} \exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}\beta)\right) (A-\mathrm{i}\beta)w - \frac{1}{\beta} \langle \psi, \xi \rangle_{\mathbb{C}^n} \phi \, d\tau \\
&= e^{\mathrm{i}t} \left[\exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}\beta)\right) w - \frac{\tau}{\beta} \langle \psi, \xi \rangle_{\mathbb{C}^n} \phi\right]_0^t \\
&= e^{\mathrm{i}t} w - \exp\left(\frac{t}{\beta}A\right) w - \frac{t}{\beta} e^{\mathrm{i}t} \langle \psi, \xi \rangle_{\mathbb{C}^n} \phi \\
&= \left(e^{\mathrm{i}t} id - e^{\frac{t}{\beta}A}\right) w - \frac{t}{\beta} e^{\mathrm{i}t} \langle \psi, \xi \rangle_{\mathbb{C}^n} \phi.
\end{aligned}$$

Here we used in particular that $\phi \in \ker(A - \mathrm{i}\beta)$ and hence

$$\exp\left(A - \mathrm{i}\beta\right) \phi = \sum_{k=0}^{\infty} \frac{1}{k!}(A - \mathrm{i}\beta)^k \phi = \phi + \underbrace{\sum_{k=1}^{\infty} \frac{1}{k!}(A - \mathrm{i}\beta)^k \phi}_{=0} = \phi.$$

Problem 29 (Simplicity (H2))

Let $\beta > 0$ and $A \in \mathbb{R}^{n\times n}$ with

$$\ker(A - \mathrm{i}\beta) = \operatorname{span}\{\phi\}, \quad \ker(A^T + \mathrm{i}\beta) = \operatorname{span}\{\psi\}, \quad \langle \psi, \phi\rangle_{\mathbb{C}^n} = 1.$$

Show that

$$\psi \cdot \phi = 0.$$

Solution

For ϕ, ψ as above we have

$$\begin{aligned} 0 &= (A^T + \mathrm{i}\beta)\psi \cdot \phi \\ &= \left\langle (A^T + \mathrm{i}\beta)\psi, \overline{\phi} \right\rangle_{\mathbb{C}^n} \\ &= \left\langle \psi, (A - \mathrm{i}\beta)\overline{\phi} \right\rangle_{\mathbb{C}^n} \\ &= \left\langle \psi, \overline{(A + \mathrm{i}\beta)\phi} \right\rangle_{\mathbb{C}^n} \\ &= \left\langle \psi, \overline{2\mathrm{i}\beta\phi} \right\rangle_{\mathbb{C}^n} \\ &= 2\mathrm{i}\beta \left\langle \psi, \overline{\phi} \right\rangle_{\mathbb{C}^n} \\ &= 2\mathrm{i}\beta \psi \cdot \phi \end{aligned}$$

and hence

$$\psi \cdot \phi = 0.$$

Problem 30 (Assumptions (H1)-(H4))

Consider the differential equation

$$-y'' = g(y', y, \lambda).$$

Under which conditions on g are the assumptions (H1)-(H4) valid for some given $\lambda_0 \in \mathbb{R}$, $\beta > 0$?

Hint: Write the differential equation as a two-dimensional dynamical system in the variable $x = (y, y')$.

Solution

We rewrite our differential equation as a two-dimensional system in the variable $x = (y, y')$, i.e.

$$x' = f(x, \lambda)$$

where

$$f(x, \lambda) = \begin{pmatrix} x_2 \\ -g(x_2, x_1, \lambda) \end{pmatrix}.$$

Step 1: (H1) iff $g \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g(0, 0, \lambda_0) = 0$.

We define $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(p, x, \lambda) \mapsto g(p, x, \lambda)$. Then we observe that $f \in C^2(\mathbb{R}^2, \mathbb{R})$ iff $g \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $f(0, \lambda_0) = 0$ iff $g(0, 0, \lambda_0) = 0$.

Step 2: (H2) iff $g_x(0, 0, \lambda_0) = \beta^2$, $g_p(0, 0, \lambda_0) = 0$ and ϕ, ψ chosen as $\phi = \begin{pmatrix} 1 \\ \mathrm{i}\beta \end{pmatrix}$ and $\psi = \frac{1}{2\beta} \begin{pmatrix} \beta \\ \mathrm{i} \end{pmatrix}$.

We calculate

$$f_x(0, \lambda_0) = \begin{pmatrix} 0 & 1 \\ -g_x(0, 0, \lambda_0) & -g_p(0, 0, \lambda_0) \end{pmatrix}$$

and hence

$$f_x(0, \lambda_0) - \mathrm{i}\beta = \begin{pmatrix} -\mathrm{i}\beta & 1 \\ -g_x(0, 0, \lambda_0) & -g_p(0, 0, \lambda_0) - \mathrm{i}\beta \end{pmatrix}$$

and

$$f_x(0, \lambda_0)^T + \mathrm{i}\beta = \begin{pmatrix} \mathrm{i}\beta & -g_x(0, 0, \lambda_0) \\ 1 & -g_p(0, 0, \lambda_0) + \mathrm{i}\beta \end{pmatrix}.$$

Thus,

$$\ker\left(f_x(0,\lambda_0)-\mathrm{i}\beta\right)=\ker\begin{pmatrix}-\mathrm{i}\beta & 1\\ -g_x(0,0,\lambda_0) & -g_p(0,0,\lambda_0)-\mathrm{i}\beta\end{pmatrix}=\ker\begin{pmatrix}-\mathrm{i}\beta & 1\\ \beta^2-g_x(0,0,\lambda_0) & -g_p(0,0,\lambda_0)\end{pmatrix}$$

and this kernel is nontrivial iff $g_p(0,0,\lambda_0)=0$ and $g_x(0,0,\lambda_0)=\beta^2$. Then

$$\ker\left(f_x(0,\lambda_0)-\mathrm{i}\beta\right)=\operatorname{span}\left\{\begin{pmatrix}1\\ \mathrm{i}\beta\end{pmatrix}\right\}.$$

Additionally we see that in this case also $\ker\left(f_x(0,\lambda_0)^T+\mathrm{i}\beta\right)$ is nontrivial as

$$\ker\left(f_x(0,\lambda_0)^T+\mathrm{i}\beta\right)=\ker\begin{pmatrix}\mathrm{i}\beta & -\beta^2\\ 1 & \mathrm{i}\beta\end{pmatrix}=\operatorname{span}\left\{\frac{1}{2\beta}\begin{pmatrix}\beta\\ \mathrm{i}\end{pmatrix}\right\}.$$

With $\phi:=\begin{pmatrix}1\\ \mathrm{i}\beta\end{pmatrix}$ and $\psi:=\frac{1}{2\beta}\begin{pmatrix}\beta\\ \mathrm{i}\end{pmatrix}$ we also have

$$\langle\phi,\psi\rangle=\phi\cdot\overline{\psi}=\begin{pmatrix}1\\ \mathrm{i}\beta\end{pmatrix}\cdot\frac{1}{2\beta}\begin{pmatrix}\beta\\ -\mathrm{i}\end{pmatrix}=1.$$

Step 3: With g chosen as above, (H3) is fulfilled.

We calculate

$$f_x(0,\lambda_0)+\mathrm{i}k\beta=\begin{pmatrix}\mathrm{i}k\beta & -\beta^2\\ 1 & \mathrm{i}k\beta\end{pmatrix}$$

and hence $\det(f_x(0,\lambda_0)+\mathrm{i}k\beta)=-k^2\beta^2+\beta^2\neq 0$ for all $k\in\mathbb{Z}$, $|k|\neq 1$. Consequently $\ker\left(f_x(0,\lambda_0)+\mathrm{i}k\beta\right)=\{0\}$.

Step 4: (H4) is fulfilled iff $g_{p\lambda}(0,0,\lambda_0)\neq 0$.

We observe

$$f_{x\lambda}(0,0,\lambda_0)=\begin{pmatrix}0 & 0\\ -g_{x\lambda}(0,0,\lambda_0) & -g_{p\lambda}(0,0,\lambda_0)\end{pmatrix}$$

and

$$\begin{aligned}&\langle f_{x\lambda}(0,0,\lambda_0)\phi,\psi\rangle_{\mathbb{C}^n}=f_{x\lambda}(0,0,\lambda_0)\phi\cdot\overline{\psi}\\ =&\begin{pmatrix}0\\ -g_{x\lambda}(0,0,\lambda_0)-\mathrm{i}\beta g_{p\lambda}(0,0,\lambda_0)\end{pmatrix}\cdot\frac{1}{2\beta}\begin{pmatrix}\beta\\ -\mathrm{i}\end{pmatrix}\\ =&\frac{\mathrm{i}}{2\beta}g_{x\lambda}(0,0,\lambda_0)-\frac{1}{2}g_{p\lambda}(0,0,\lambda_0).\end{aligned}$$

Finally,

$$\operatorname{Re}\left(\langle f_{x\lambda}(0,0,\lambda_0)\phi,\psi\rangle_{\mathbb{C}^n}\right)=-\frac{1}{2}g_{p\lambda}(0,0,\lambda_0)\neq 0$$

iff $g_{p\lambda}(0,0,\lambda_0)\neq 0$ which closes the proof.