Bifurcation Theory
Problem Sheet 12

Problem 31 (Gradient systems cannot have periodic solutions)
A gradient system is a dynamical system of the form
\[ x' = -\nabla_x V(x, \lambda) \]
for a given function \( V \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}) \).

(a) Prove that gradient systems cannot have non-constant periodic solutions.

(b) Let \( V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}) \). Show that Hopf bifurcation cannot occur in gradient systems.

(c) Consider the dynamical system
\[
\begin{align*}
    x_1' &= 2 - x_1 - \lambda x_2^2, \\
    x_2' &= 5 - x_2 - 2\lambda x_1 x_2.
\end{align*}
\]
Are there periodic solutions?

Problem 32 (Langford-System)
Determine the Hopf bifurcation points \((0, \lambda_0) \in \mathbb{R}^3 \times \mathbb{R}\) of the nonlinear system
\[
\begin{align*}
    x_1' &= (\lambda - 1)x_1 - x_2 + x_1 x_3, \\
    x_2' &= x_1 + (\lambda - 1)x_2 + x_2 x_3, \\
    x_3' &= \lambda x_3 - (x_1^2 + x_2^2 + x_3^2).
\end{align*}
\]