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Bifurcation Theory

Solutions to Problem Sheet 12

Problem 31 (Gradient systems cannot have periodic solutions)

A gradient system is a dynamical system of the form

$$x' = -\nabla_x V(x, \lambda)$$

for a given function $V \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$.

(a) Prove that gradient systems cannot have non-constant periodic solutions.

(b) Let $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$. Show that Hopf bifurcation cannot occur in gradient systems.

(c) Consider the dynamical system

$$\begin{cases} x_1' = 2 - x_1 - \lambda x_2^2, \\ x_2' = 5 - x_2 - 2\lambda x_1 x_2. \end{cases}$$

Are there periodic solutions?

Solution

(a) We suppose that there is a non-constant periodic solution. Then we have

$$\begin{aligned} 0 &= V(x(T), \lambda) - V(x(0), \lambda) \\ &= \int_0^T \frac{d}{dt} V(x(t), \lambda)\, dt \\ &= \int_0^T \nabla_x V(x(t), \lambda) \cdot x'(t)\, dt \\ &= \int_0^T - \|x'(t)\|^2 \, dt < 0, \end{aligned}$$

unless $x' \equiv 0$, a contradiction.

(b) Define $f(x,\lambda) = -\nabla_x V(x,\lambda)$. Then we have

$$\frac{\partial}{\partial x_i} f(x,\lambda) = \frac{\partial}{\partial x_i}(-\nabla_x V(x,\lambda)).$$

Hence, we observe

$$f_x(x,\lambda) = -H_V(x,\lambda),$$

where H_V denotes the Hessian. Thus,

$$f_x(0,\lambda_0) = -H_V(0,\lambda_0)$$

is symmetric and has only real-valued eigenvalues. Hence the simplicity condition (H2) can not be fulfilled and Hopf bifurcation is not possible.

(c) Define $V : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, $V(x,\lambda) := -2x_1 - 5x_2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \lambda x_1 x_2^2$. Then we calculate

$$\nabla_x V(x,\lambda) = \begin{pmatrix} -2 + x_1 + \lambda x_2^2 \\ -5 + x_2 + 2\lambda x_1 x_2. \end{pmatrix}$$

And hence

$$x' = -\nabla_x V(x,\lambda),$$

which means by part (a) that there is no periodic solution.

Problem 32 (Langford-System)

Determine the Hopf bifurcation points $(0, \lambda_0) \in \mathbb{R}^3 \times \mathbb{R}$ of the nonlinear system

$$\begin{cases} x_1' = (\lambda - 1)x_1 - x_2 + x_1 x_3, \\ x_2' = x_1 + (\lambda - 1)x_2 + x_2 x_3, \\ x_3' = \lambda x_3 - (x_1^2 + x_2^2 + x_3^2). \end{cases}$$

Solution

Assertion: $(0, \lambda_0)$ *is a Hopf bifurcation point if and only if* $\lambda_0 = 1$.

We define

$$f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3, \quad f(x, \lambda) := \begin{pmatrix} (\lambda - 1)x_1 - x_2 + x_1 x_3 \\ x_1 + (\lambda - 1)x_2 + x_2 x_3 \\ \lambda x_3 - (x_1^2 + x_2^2 + x_3^2) \end{pmatrix}$$

Then $f(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$ and $f \in C^2(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ with

$$f_x(x, \lambda) = \begin{pmatrix} \lambda - 1 + x_3 & -1 & x_1 \\ 1 & \lambda - 1 + x_3 & x_2 \\ -2x_1 & -2x_2 & \lambda - 2x_3 \end{pmatrix}, \quad f_{x\lambda}(x, \lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = id$$

for $x_1, x_2, x_3, \lambda \in \mathbb{R}$, and in particular, for $\lambda_0 \in \mathbb{R}$,

$$f_x(0, \lambda_0) = \begin{pmatrix} \lambda_0 - 1 & -1 & 0 \\ 1 & \lambda_0 - 1 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix}, \quad f_{x\lambda}(0, \lambda_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = id$$

The eigenvalues of $f_x(0, \lambda_0)$ are given by λ_0 and $\lambda_0 - 1 \pm \mathrm{i}$ and the eigenvalues of $f_x(0, \lambda_0)^T$ are also λ_0 and $\lambda_0 - 1 \pm \mathrm{i}$. Hence, simplicity (H2) tells us that Hopf bifurcation can only occur for $\lambda_0 = 1$. (Here $\beta = 1$.) Then,

$$\ker(f_x(0, \lambda_0) - \mathrm{i}) = \operatorname{span} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \right\}$$

and

$$\ker(f_x(0, \lambda_0)^T + \mathrm{i}) = \operatorname{span} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \right\}.$$

With $\phi := \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$ and $\psi := \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$ we then have

$$\langle \psi, \phi \rangle_{\mathbb{C}^n} = \psi \cdot \overline{\phi} = 1$$

and hence (H2) is fulfilled. Additionally we have for all $k \in \mathbb{Z}$ with $|k| \neq 1$

$$f_x(0, \lambda_0) - \mathrm{i}k = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i}k & -1 & 0 \\ 1 & -\mathrm{i}k & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

i.e. $\det(f_x(0, \lambda_0) - \mathrm{i}k) = (-\mathrm{i}k)^2 + 1 = -k^2 + 1 \neq 0$ which shows

$$\ker(f_x(0, \lambda_0) - \mathrm{i}k) = \{0\}$$

and thus proves the nonresonance condition (H3). Finally, we oberserve

$$\mathrm{Re}(\langle f_{x\lambda}(0, \lambda_0)\phi, \psi \rangle_{\mathbb{C}^n}) = \mathrm{Re}(\langle \phi, \psi \rangle_{\mathbb{C}^n}) = 1 \neq 0$$

which is exactly the transversality condition (H4) and closes the proof.