These lecture notes are work in progress.
Version: February 17, 2023

Contents

1 Introduction 2

2 Energy method and bifurcation for ODEs 7

3 Calculus in Banach spaces 14

4 Bifurcation from a simple eigenvalue 25

5 Bifurcation from Infinity 34

6 Digression: The Ljapunov-Schmidt reduction 37

7 Hopf Bifurcation 39

Bifurcation Theory

Rainer Mandel

Winter term 2022/23

1 Introduction

Suppose X, Z are real Banach spaces and $F : X \times \mathbb{R} \to \mathbb{Z}$ is a continuous function satisfying

$$F(0,\lambda) = 0 \qquad \text{for all } \lambda \in \mathbb{R}.$$

Hence, for any given $\lambda \in \mathbb{R}$, there is a "trivial" solution $x = 0$. The scope of bifurcation theory is to find necessary conditions as well as sufficient conditions for the existence of **nontrivial** solutions $(x,\lambda) \in X \setminus \{0\} \times \mathbb{R}$ of the equation $F(x,\lambda) = 0$ near the "trivial solution family" $\{(0,\lambda) : \lambda \in \mathbb{R}\}$.

Here are some examples:

(i) Consider the elementary equation

$$\lambda x - x^3 = 0 \qquad (x \in \mathbb{R}, \lambda \in \mathbb{R})$$

This fits into the above-mentioned framework by choosing $F : X \times \mathbb{R} \to Z, (x,\lambda) \mapsto \lambda x - x^3$ for $X = Z = \mathbb{R}$. There are

- trivial solutions $(0,\lambda)$ for $\lambda \in \mathbb{R}$
- nontrivial solutions $(\pm\sqrt{\lambda}, \lambda)$ for $\lambda > 0$

This situation can be visualized in a so-called bifurcation diagram.

(ii) Consider another elementary equation

$$\lambda x - Ax = 0 \qquad (x \in \mathbb{R}^n, \lambda \in \mathbb{R})$$

where $A \in \mathbb{R}^{n\times n}$ is a given square matrix. Again, this can be written in abstract form as $F(x,\lambda) = 0$ for $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $(x,\lambda) \mapsto \lambda x - Ax$. As before, there are

- trivial solutions $(0,\lambda)$ for $\lambda \in \mathbb{R}$

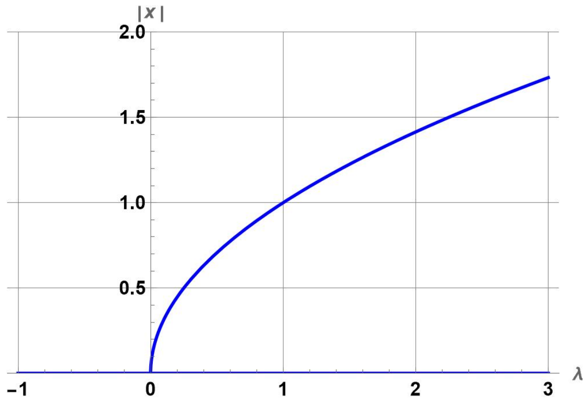

Figure 1: Bifurcation diagram for (i)

– nontrivial solutions $(t\varphi, \lambda_0)$ for $t \in \mathbb{R}$ where (φ, λ_0) is an eigenpair of A. We will see in this course that this is not a coincidence! The eigenvalues of linear operators play a decisive role in bifurcation theory because the Implicit Function Theorem (IFT) is not applicable in these points (as we shall see later). The corresponding bifurcation diagram is given as follows:

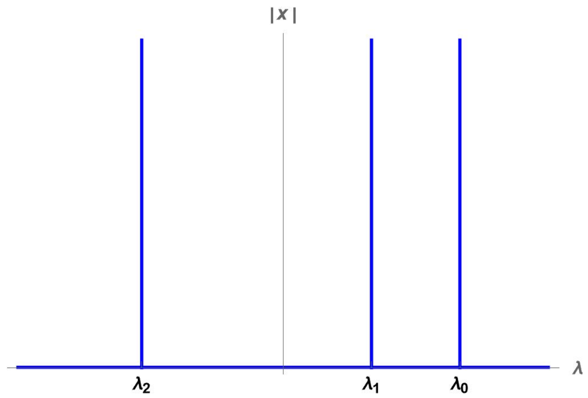

Figure 2: Bifurcation diagram for (ii)

(iii) The following example shows that bifurcation does not always occur. Consider for instance

$$\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1^3 \end{pmatrix} \qquad (x = (x_1, x_2) \in \mathbb{R}^2, \lambda \in \mathbb{R})$$

Once more, there is a trivial solution family consisting of $(0, \lambda)$ for all $\lambda \in \mathbb{R}$, but there are no(!) nontrivial solutions. So bifurcation does not occur. This is proved as follows: if (x_1, x_2, λ) is a solution then

$$0 = \lambda(x_2 + \lambda x_1) = \lambda x_2 + \lambda^2 x_1 = x_1^3 + \lambda^2 x_1 = x_1(x_1^2 + \lambda^2).$$

In any case we find $x_1 = 0$ and hence $x_2 = 0$ as well. So any solution must be trivial.

(iv) We now analyze a more challenging ODE problem - the pendulum equation. The trajectory $t \mapsto (x(t), y(t))$ of the appended mass m is given by

$$x(t) = l\sin(\theta(t)), \qquad y(t) = l(1 - \cos(\theta(t)))$$

where l denotes the length of the pendulum. The question is how to describe the angle θ. We have an energy balance: the sum of kinetic energy and potential energy (g is the gravitational constant on earth) is constant, i.e.

$$\begin{aligned} E'(t) = 0 \quad \text{where} \quad E(t) &= \frac{m}{2}(x'(t)^2 + y'(t)^2) + mgy(t) \\ &= \frac{m}{2}l^2\theta'(t)^2 + mgl(1 - \cos(\theta(t))). \end{aligned}$$

Here, any frictional effect is neglected. If we denote by T the return time to rest position, we find the following model for $\lambda := \frac{g}{l}$

$$\theta''(t) + \lambda\sin(\theta(t)) = 0, \qquad \theta(0) = \theta(T) = 0. \tag{1.1}$$

This ODE boundary value problem can also be written as $F(x, \lambda) = 0$ for $x \in X$ and $\lambda \in \mathbb{R}$. One possible choice is $X = C^2([0,T];\mathbb{R}), Z = C([0,T];\mathbb{R}) \times \mathbb{R}^2$ and

$$F(\theta, \lambda) := \big(\theta'' + \lambda\sin(\theta), \theta(0), \theta(T)\big).$$

Again, one can draw a bifurcation diagram that helps to visualize the situation: In fact, we will prove later that there are infinitely many bifurcation

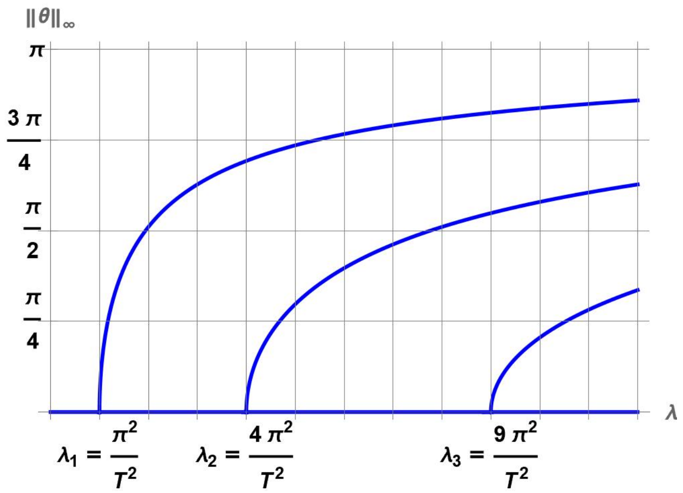

Figure 3: Bifurcation diagram for (iv)

points $(0, \frac{j^2\pi^2}{T^2})$ belonging to the trivial solution family $\{(0, \lambda) : \lambda \in \mathbb{R}\}$. The emanating curves $\mathcal{C}_j$ turn out to be mutually disjoint and each point in that diagram corresponds to two solutions of the pendulum boundary value problem (1.1) with $j-1$ interior zeros[1]. We shall see this later.

Definition 1.1. *Let X, Z be Banach spaces and $F \in C(X \times \mathbb{R},\ Z)$ with $F(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Then $(0,\lambda_0)$ is called a bifurcation point for the equation $F(x,\lambda) = 0$ if there is a sequence of solutions (x_n,λ_n) such that $x_n \neq 0$ and $(x_n,\lambda_n) \to (0,\lambda_0)$ as $n \to \infty$.*

Remark 1.2.

(a) Very often $X = Z$ are Hilbert spaces and F is much smoother.

(b) Assume there is another known curve $\{(\bar{x}(\lambda),\lambda) : \lambda \in \mathbb{R}\}$ with the property $F(\bar{x}(\lambda),\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Then one may study bifurcation from this known solution curve by studying $G(x,\lambda) := F(x+\bar{x}(\lambda),\lambda)$.

(c) In many applications the bifurcating solutions in fact lie on a curve in $X \times \mathbb{R}$ that intersects the trivial solution curve transversally. This is true in all examples discussed above. There are, however, examples where this is no longer true.

End of Lec01

The following result shows that without any structural assumptions on the function F one may encounter very strange bifurcation phenomena when we choose $Y = X \times \mathbb{R}$ in the next lemma. Since we do not have calculus in Banach spaces at our disposal yet, we formulate the following result for Euclidean spaces Y only[2].

Lemma 1.3. *Let $Y := \mathbb{R}^M$ and let Z be a nonempty normed space. Then every closed subset $A \subset Y$ is the zero set of a smooth function $F \in C^\infty(Y;Z)$.*

Proof:
Without loss of generality assume $\emptyset \subsetneq A \subsetneq Y$. We fix a smooth function $\psi : \mathbb{R} \to [0,1]$ such that $\psi(z) = 0$ for $z \geq \frac{1}{2}$ and $\psi(z) = 1$ for $z \leq \frac{1}{4}$. For any given $y \in A^c := Y \setminus A$ define $\phi_y(\tilde{y}) := \psi(|\tilde{y}-y|/\operatorname{dist}(y,A))$. Then $\psi \in C^\infty(Y;[0,1])$ and

$$\phi_y(\tilde{y}) = 1 \quad \text{for } |\tilde{y}-y| \leq \frac{1}{4}\operatorname{dist}(y,A), \qquad \phi_y(\tilde{y}) = 0 \quad \text{for } |\tilde{y}-y| > \frac{1}{2}\operatorname{dist}(y,A).$$

Define the open neighbourhood of y given by $U_y := \{\tilde{y} \in Y : \phi_y(\tilde{y}) > 0\} \subset A^c$. If we denote by $D \subset Y$ a countable dense set then

$$A^c = \bigcup_{Y \in D \cap A^c} U_y = \bigcup_{i=1}^{\infty} U_{y_i} \qquad \text{where } D \cap A^c = \{y_1, y_2, \ldots\}. \tag{1.2}$$

[1] One also says that the solutions have exactly j nodal domains. Think of functions like $t \mapsto \pm\sin(j\pi t/T)$ to get an idea of the shape of these two solutions.

[2] One may check that all assumptions about Y are that Y is separable and equipped with a norm $\|\cdot\|$ such that for any closed and bounded subset $B \subset Y \setminus \{0\}$ the map $y \mapsto \|y\| \in C^\infty(B;\mathbb{R})$ with bounded derivatives on B. This holds for L^p-spaces with $1 < p < \infty$, or Hilbert spaces.

Indeed:

- "$\supset$" This follows from $U_y \subset A^c$ for all $y \in A^c \cap D$, in particular $U_{y_i} \subset A^c$ for all $i \in \mathbb{N}$.
- "$\subset$" Let $y \in A^c$. Choose $y_j \in D \cap A^c$ such that $|y - y_j| \leq \frac{1}{5} \operatorname{dist}(y, A)$. Then we have for all $a \in A$
$$|y - y_j| \leq \frac{1}{5}|y - a| \leq \frac{1}{5}|y - y_j| + \frac{1}{5}|y_j - a|$$
and thus $|y - y_j| \leq \frac{1}{4} \operatorname{dist}(y_j, A)$. So $\phi_{y_j}(y) = 1$ and thus
$$y \in U_{y_j} \subset \bigcup_{i=1}^{\infty} U_{y_i}$$

In view of (1.2) let us define[3] for some $z \in Z \setminus \{0\}$

$$F(y) := \left(\sum_{i=1}^{\infty} c_i \phi_{y_i}(y)\right) z \quad \text{where } c_i := \frac{1}{2^i \max\{|\partial^\alpha \phi_{y_i}(y)| : y \in Y, \alpha \in \mathbb{N}_0^M, |\alpha| \leq i\}}.$$

This series indeed converges and F is continuous by Weierstrass' M-test. Indeed,

$$\left\| \sum_{i=1}^{N} c_i \phi_{y_i}(y) - \sum_{i=1}^{M} c_i \phi_{y_i}(y) \right\| \leq \sum_{i=M}^{N} \|c_i \phi_{y_i}(y) z\| \leq \sum_{i=M}^{N} 2^{-i} \|z\| \to 0 \quad \text{as } N \geq M \to \infty.$$

Similarly, we can prove $F \in C^\infty(Y; Z)$ because we have for $N \geq M \geq |\alpha|$

$$\begin{aligned} \left\| \partial^\alpha \left(\sum_{i=1}^{N} c_i \phi_{y_i}(y) z \right) - \partial^\alpha \left(\sum_{i=1}^{M} c_i \phi_{y_i}(y) z \right) \right\| &\leq \sum_{i=M}^{N} \|c_i (\partial^\alpha \phi_{y_i}(y)) z\| \\ &\leq \sum_{i=M}^{N} 2^{-i} \|z\| \to 0 \quad \text{as } N \geq M \to \infty. \end{aligned}$$

So F is well-defined, smooth and we have

$$\begin{aligned} F(y) = 0 \quad &\Leftrightarrow \quad \sum_{i=1}^{\infty} c_i \phi_{y_i}(y) = 0 \\ &\Leftrightarrow \quad \phi_{y_i}(y) = 0 \quad \text{for all } i \in \mathbb{N} \\ &\Leftrightarrow \quad y \notin U_{y_i} \quad \text{for all } i \in \mathbb{N} \\ &\Leftrightarrow \quad y \in \left(\bigcup_{i \in \mathbb{N}} U_{y_i} \right)^c \\ &\overset{(1.2)}{\Leftrightarrow} \quad y \in A. \end{aligned}$$

This is all we had to prove. □

[3]The whole difficulty is to show that F_i is well-defined, i.e., that this maximum exists or at least that there is a finite upper bound for all these derivatives.

2 Energy method and bifurcation for ODEs

We consider again the pendulum equation

$$\theta''(t) + \lambda \sin(\theta(t)) = 0 \quad \text{on } (0,T), \qquad \theta(0) = \theta(T) = 0. \tag{2.1}$$

First we are interested in solutions of this problem that are positive on $(0,T)$. This corresponds to one deflection of the pendulum to the right and its return to the rest position within time T. We have the following facts about solutions θ to (2.1):

- If θ is positive, then $-\theta$ is a negative solution of (2.1).
- θ is even about zeros of θ' and it is odd about zeros of θ. In other words[4],

$$\begin{aligned} \theta'(t^*) = 0 \quad &\Rightarrow \quad \theta(t^* + t) = \theta(t^* - t) \quad \forall t \in \mathbb{R} \\ \theta(t^*) = 0 \quad &\Rightarrow \quad \theta(t^* + t) = -\theta(t^* - t) \quad \forall t \in \mathbb{R} \end{aligned}$$

  In particular, θ extends to a $2T$−periodic solutions on $\mathbb{R}$ by odd reflection about the points $kT, k \in \mathbb{Z}$.
- θ is infinitely many times differentiable on $[0,T]$.

The energy method allows to solve equations like (2.1) semi-explicitly. The basica idea is to multiply (2.1) with θ', to integrate the resulting equation[5] and then to solve the new autonomous ODE of first order. Let's do this for positive solutions of the above model problem with maximum at t^*, so $\alpha := \theta(t^*) = \|\theta\|_\infty$. To simplify the following we pretend to already know:

$$t^* = T/2. \qquad \theta' > 0 \text{ on } (0,T/2),\ \theta' < 0 \text{ on } (T/2,T), \qquad \lambda \geq 0.$$

With this we get:

$$\begin{aligned} & \theta''(t)\theta'(t) + \lambda \sin(\theta(t))\theta'(t) = 0 \qquad \forall t \in \mathbb{R} \\ \Longrightarrow \quad & \frac{1}{2}\theta'(t)^2 - \lambda \cos(\theta(t)) = \mu \qquad \forall t \in \mathbb{R} \quad \text{where } \mu \in \mathbb{R} \\ \Longrightarrow \quad & \theta'(t)^2 = 2\lambda(\cos(\theta(t)) - \cos(\theta(t^*))) \qquad \forall t \in \mathbb{R} \\ \Longrightarrow \quad & \frac{\theta'(t)}{\sqrt{|\cos(\theta(t)) - \cos(\alpha)|}} = \sqrt{2\lambda} \quad \text{on } (0,T/2) \\ & \frac{\theta'(t)}{\sqrt{|\cos(\theta(t)) - \cos(\alpha)|}} = -\sqrt{2\lambda} \quad \text{on } (T/2,T) \\ \Longrightarrow \quad & \int_{\theta(0)}^{\theta(t)} \frac{1}{\sqrt{|\cos(z) - \cos(\alpha)|}}\, dz = \sqrt{2\lambda} t \quad \text{on } (0,T/2) \end{aligned}$$

[4] This follows from the unique solvability of the initial value problem (Picard-Lindelöf Theorem) at the point t^*. Fill in the details!

[5] in order to get a "first integral" of the problem

$$\int_{\theta(t)}^{\theta(T)} \frac{1}{\sqrt{|\cos(z) - \cos(\alpha)|}}\, dz = -\sqrt{2\lambda}(T-t) \quad \text{on } (T/2, T)$$

$$\overset{\theta(0)=\theta(T)=0}{\Longrightarrow} \quad \int_{0}^{\theta(t)} \frac{1}{\sqrt{|\cos(z) - \cos(\alpha)|}}\, dz = \sqrt{2\lambda}t \text{ and } \theta(t) = \theta(T-t) \quad \text{for } 0 < t < T/2.$$

This provides an implicit solution formula for $\theta(t)$. In view of $\theta(T/2) = \alpha$ we moreover have

$$\int_{0}^{\alpha} \frac{1}{\sqrt{|\cos(z) - \cos(\alpha)|}}\, dz = \sqrt{\frac{\lambda}{2}}\,T. \tag{2.2}$$

Summary: A positive smooth solution θ with maximum $\alpha = \|\theta\|_\infty$ can be found at λ provided that (2.2) holds. In this case, it is given by $\theta(t) = \theta(T-t)$ as well as

$$\int_{0}^{\theta(t)} \frac{1}{\sqrt{|\cos(z) - \cos(\alpha)|}}\, dz = \sqrt{2\lambda}t \qquad \text{for } 0 < t < T/2.$$

Hence, positive solutions bifurcate from $(0, \lambda_0)$ in $C([0,T];\mathbb{R})$ (or $L^\infty([0,T];\mathbb{R})$) where

$$\lambda_0 = \lim_{\alpha \to 0} \frac{2}{T^2} \left(\int_{0}^{\alpha} \frac{1}{\sqrt{|\cos(z) - \cos(\alpha)|}}\, dz \right)^2 = \frac{2}{T^2} \cdot \frac{\pi^2}{2} = \frac{\pi^2}{T^2},$$

as we shall prove below.

Definition 2.1. *A function $u : [0,T] \to \mathbb{R}$ is called j-nodal if it has exactly j zeros in $(0,T)$ and all of them are simple, i.e., the derivative does not vanish at these points.*

Given the symmetries of (2.1) the construction of j-nodal solutions on $[0,T]$ amounts to finding positive solutions on $[0, T/(j+1)]$ by odd reflection. In this way, we find that j-nodal solutions of (2.1) bifurcate from the trivial solution family at $(0, \lambda_j)$ where

$$\lambda_j = \frac{(j+1)^2 \pi^2}{T^2}.$$

End of Lec02

We now generalizing these observations to boundary value problems of the form

$$u'' + g(u, \lambda) = 0 \quad \text{on } (0,T), \qquad u(0) = u(T) = 0. \tag{2.3}$$

First we investigate the general shape of solutions.

Proposition 2.2. *Let $\alpha_0 \in (0, \infty)$ and assume that $g \in C^1((-\alpha_0, \alpha_0) \times \mathbb{R}; \mathbb{R})$ satisfies*

$$g(z, \lambda) = -g(-z, \lambda) > 0 \quad \textit{whenever } 0 < z < \alpha_0, \lambda \in \mathbb{R}.$$

Let u be a j-nodal solution of the boundary value problem (2.3) *with* $\|u\|_\infty = \alpha \in (0, \alpha_0)$. *Then u extends uniquely to a T_j-antiperiodic solution of the ODE on $\mathbb{R}$ via*

$$u(x) = u(T_j - x) = -u(T_j + x) \quad \forall x \in \mathbb{R} \quad \text{where } T_j := \frac{T}{j+1}. \tag{2.4}$$

Moreover, we have $u'(x) = 0$ if and only if $|u(x)| = \alpha$ if and only if $x = \frac{(2k+1)T}{2(j+1)}, k \in \mathbb{Z}$.

Proof:
We will w.l.o.g. assume $u'(0) > 0$, otherwise[6] consider the solution $x \mapsto -u(x)$. We shall use that the assumption on g implies that

(i) u is odd about its zeros and

(ii) u is even about its critical points.

Let $x^* \in (0, T]$ be smallest possible such that u is positive on the interval $(0, x^*)$ and $u(x^*) = 0$. Such an x^* exists because of $u'(0) > 0$ and $u(0) = u(T) = 0$. In view of $u''(x) = -g(u(x), \lambda) < 0$, u is strictly concave on that interval, so u has a unique maximizer on $[0, x^*]$. By uniqueness and (ii), this maximum can only be attained at $x^*/2$. So we know that

$$u(0) = 0, \quad u' > 0 \text{ on } (0, x^*/2), \quad u'(x^*/2) = 0, \quad u' < 0 \text{ on } (x^*/2, x^*), \quad u(x^*) = 0.$$

In view of (i) these properties carry over to the intervals $(kx^*, (k+1)x^*)$ with $k \in \mathbb{Z}$. Given that u has precisely j zeros in $(0, T)$ and $u(0) = u(T) = 0$ (by assumption) we obtain $x^* = \frac{T}{j+1}$ and (i),(ii) imply (2.4). Furthermore, we have shown that the local extrema at $(2k+1)x^*/2, k \in \mathbb{Z}$ are the only critical points of u and the only points $x \in \mathbb{R}$ satisfying $|u(x)| = \alpha$. □

In the following we write $G(z, \lambda) := \int_0^z g(s, \lambda)\, ds$.

Theorem 2.3. *Let $\alpha_0 \in (0, \infty)$ and assume that $g \in C^1((-\alpha_0, \alpha_0) \times \mathbb{R}; \mathbb{R})$ satisfies*

$$g(z, \lambda) = -g(-z, \lambda) > 0 \quad \text{whenever } 0 < z < \alpha_0, \lambda \in \mathbb{R}.$$

Then the boundary value problem (2.3) *has a j-nodal solution $u \in L^\infty(0, T)$ with $\|u\|_\infty = \alpha \in (0, \alpha_0)$ if and only if*

$$\int_0^\alpha \frac{1}{\sqrt{G(\alpha, \lambda) - G(z, \lambda)}}\, dz = \frac{T}{\sqrt{2}(j+1)}. \tag{2.5}$$

[6]Note that $u'(0) = 0$ is impossible because the Picard-Lindelöf Theorem would imply $u \equiv 0$. Here we use that the nonlinearity is continuously differentiable and thus locally Lipschitz continuous.

In this case we have $u = +u_\alpha$ or $u = -u_\alpha$ where $u_\alpha(x) \in [0,\alpha]$ is uniquely determined by

$$\begin{aligned} &\int_0^{u_\alpha(x)} \frac{1}{\sqrt{G(\alpha,\lambda) - G(z,\lambda)}}\, dz = \sqrt{2}x \quad \textit{for } 0 \le x \le T_j/2 \qquad \textit{and} \\ &u_\alpha(x) = u_\alpha(T_j - x) = -u_\alpha(T_j + x) \quad \forall x \in \mathbb{R} \quad \textit{where } T_j := \frac{T}{j+1}. \end{aligned} \tag{2.6}$$

In particular, $\pm u_\alpha$ are the only j-nodal solutions with L^∞-norm α.

Proof:
We first assume that u is a j-nodal solution with $\|u\|_\infty = \alpha$. We assume w.l.o.g. $u'(0) > 0$ and it remains to deduce $u = u_\alpha$. Proposition 2.2 implies $u' > 0$ on $(0, T_j/2)$ and $u(T_j/2) = \|u\|_\infty = \alpha$. So multiplying the ODE with $2u'$ and integrating the resulting expression from 0 to $x \in [0, T_j/2]$ we get

$$u'(x)^2 = 2(G(\alpha,\lambda) - G(u(x),\lambda)) \quad \text{for } 0 \le x \le T_j/2.$$

Since u is increasing on $(0, T_j/2)$ we get

$$\int_0^{u(x)} \frac{1}{\sqrt{G(\alpha,\lambda) - G(z,\lambda)}}\, dz = \sqrt{2}x \quad \text{for } 0 \le x \le T_j/2.$$

This proves $u = u_\alpha$ where u_α is given by the first line in (2.6). Plugging $x = T_j/2$ and $\|u_\alpha\| = \alpha$ into (2.6) we get (2.5).

Next we show that (2.5),(2.6) indeed define j-nodal solutions $\pm u_\alpha$ with L^∞-norm α. Indeed, differentiating (2.6) shows $u_\alpha' > 0$ on $(0, T_j/2)$ as well as $u_\alpha'(x)^2 = 2(G(\alpha,\lambda) - G(u(x),\lambda))$ on $(0, T_j/2)$. Differentiating this once more and dividing the resulting equation by u_α' (which is positive) implies that u solves the ODE from (2.3). Moreover, (2.5) ,(2.6) imply $u_\alpha(0) = 0$ and $\|u_\alpha\|_\infty = u_\alpha(T_j/2) = \alpha$. □

Remark 2.4.

(i) For autonomous nonlinearities g that are not odd one may deduce similar, but more complicated formulas. Roughly speaking, in that general case there are two α-dependent positive profiles $P1, P2$ with lengths $L1, L2$ between consecutive zeros such that the solutions of (2.3) look like $[P1, -P2, P1, -P2, \ldots]$ in case $u'(0) > 0$ or like $[-P2, P1, -P2, P1, \ldots]$ in case $u'(0) < 0$. The analysis in the case of odd nonlinearities is much simpler because of $P1 = P2$.

(ii) If $g(z,\lambda)z \le 0$ for $0 < |z| < \alpha_0$ solutions of (2.3) do not exist. Indeed, "testing" the equation with u gives

$$\int_0^T u'(x)^2\, dx = \int_0^T (u'u)'(x) - u''(x)u(x)\, dx$$

$$
\begin{aligned}
&= (uu')(T) - (uu')(0) + \int_0^T g(u(x),\lambda)u(x)\,dx \\
&\leq 0.
\end{aligned}
$$

So u is constant and thus $u \equiv 0$.

The solution theory is thus reduced to solving the "algebraic equation" (2.5). We find bifurcation results from the trivial solution curve by investigating the limit $\alpha \searrow 0^+$ in that equation.

Corollary 2.5. *Let $\alpha_0 \in (0,\infty)$ and assume that $g \in C^1((-\alpha_0,\alpha_0)\times\mathbb{R};\mathbb{R})$ satisfies*

$$g(z,\lambda) = -g(-z,\lambda) > 0 \quad \textit{whenever } 0 < z < \alpha_0, \lambda \in \mathbb{R}.$$

(i) If j-nodal solutions of (2.3) bifurcate from the trivial solution curve at the point λ then $g_z(0,\lambda) = \frac{\pi^2(j+1)^2}{T^2}$. Moreover, these solutions satisfy

$$\frac{u_\alpha(x)}{\alpha} \to \sin(\pi(j+1)x/T) \quad \textit{as } \alpha \to 0 \quad \textit{uniformly on } [0,T].$$

(ii) If $g_z(0,\lambda) = \frac{\pi^2(j+1)^2}{T^2}$ and $\mu \mapsto g_z(0,\mu)$ is strictly monotone near λ, then indeed j-nodal solutions of (2.3) bifurcate from $(0,\lambda)$.

Proof:
To prove (i) we first assume that j-nodal solutions bifurcate from $(0,\lambda)$ in $L^\infty(0,T)\times\mathbb{R}$. Then Theorem 2.3 implies that there is a sequence (α_n,λ_n) with $\alpha_n \neq 0, \alpha_n \searrow 0, \lambda_n \to \lambda$ such that

$$\int_0^1 \frac{1}{\sqrt{\frac{G(\alpha_n,\lambda_n)-G(\alpha_n s,\lambda_n)}{\alpha_n^2}}}\,ds = \frac{T_j}{\sqrt{2}} \qquad \forall n \in \mathbb{N}. \tag{2.7}$$

As $n \to \infty$ we have

$$
\begin{aligned}
\frac{G(\alpha_n,\lambda_n)-G(\alpha_n s,\lambda_n)}{\alpha_n^2} &= \int_s^1 \frac{g(\tau\alpha_n,\lambda_n)}{\alpha_n}\,d\tau \\
&= \int_s^1 \int_0^\tau g_z(\mu\alpha_n,\lambda_n)\,d\mu\,d\tau \\
&= (g_z(0,\lambda)+o(1)) \int_s^1 \int_0^\tau 1\,d\mu\,d\tau \\
&= (g_z(0,\lambda)+o(1))\frac{1-s^2}{2}.
\end{aligned}
$$

As a consequence, $g_z(0,\lambda)$ must be positive[7]. Since $s \mapsto (1-s^2)^{-1/2}$ is integrable, (2.7)

[7] If it was negative, then the square root would not have been well-defined, which would contradict $g(z,\lambda) = -g(-z,\lambda) > 0$ for $0 < z < \alpha_0, \lambda \in \mathbb{R}$. If it was zero, then the left hand side in (2.7) would tend to $+\infty$ so that the equation cannot be valid.

and the Dominated Convergence Theorem give

$$\frac{T_j}{\sqrt{2}} = \int_0^1 \frac{1}{\sqrt{g_z(0,\lambda)\frac{1-s^2}{2}}}\,ds = \frac{\pi}{\sqrt{2}\sqrt{g_z(0,\lambda)}}.$$

This implies

$$g_z(0,\lambda) = \pi^2 T_j^{-2} = \frac{\pi^2(j+1)^2}{T^2}.$$

Moreover, an analogous computation gives in view of (2.6) for $\phi_\alpha(x) := \alpha^{-1}u_\alpha(x)$

$$\begin{aligned}
\sqrt{2}x &= \int_0^{u_\alpha(x)} \frac{1}{\sqrt{G(\alpha,\lambda)-G(z,\lambda)}}\,dz \\
&= \int_0^{\phi_\alpha(x)} \frac{1}{\sqrt{\frac{G(\alpha,\lambda)-G(\alpha s,\lambda))}{\alpha^2}}}\,ds \\
&= \int_0^{\phi_\alpha(x)} \frac{1}{\sqrt{(g_z(0,\lambda)+o(1))\frac{1-s^2}{2}}}\,ds \\
&= \frac{\sqrt{2}}{\sqrt{g_z(0,\lambda)+o(1)}} \int_0^{\phi_\alpha(x)} (1-s^2)^{-1/2}\,ds + o(1) \\
&= \left(\frac{\sqrt{2}T}{\pi(j+1)} + o(1)\right)\arcsin(\phi_\alpha(x)) \qquad \text{as } \alpha \to 0.
\end{aligned}$$

This proves (i).

To prove (ii) we have to ensure that (2.5) has a sequence of solutions (α_n, λ_n) such that $\alpha_n \neq 0, \alpha_n \to 0, \lambda_n \to \lambda$. W.l.o.g. assume $g_z(0,\lambda+\varepsilon) > g_z(0,\lambda) > g_z(0,\lambda-\varepsilon)$. As above one proves for all sufficiently small $\alpha > 0$

$$\begin{aligned}
\frac{G(\alpha,\lambda+\varepsilon)-G(\alpha s,\lambda+\varepsilon)}{\alpha^2}\frac{2}{1-s^2} &= g_z(0,\lambda+\varepsilon)+o(1) \\
&> g_z(0,\lambda) \\
&> g_z(0,\lambda-\varepsilon)+o(1) \\
&= \frac{G(\alpha,\lambda-\varepsilon)-G(\alpha s,\lambda-\varepsilon)}{\alpha^2}\frac{2}{1-s^2},
\end{aligned}$$

and thus

$$\int_0^1 \frac{1}{\sqrt{\frac{G(\alpha,\lambda+\varepsilon)-G(\alpha s,\lambda+\varepsilon)}{\alpha^2}}}\,ds < \frac{T_j}{\sqrt{2}} < \int_0^1 \frac{1}{\sqrt{\frac{G(\alpha,\lambda+\varepsilon)-G(\alpha s,\lambda+\varepsilon)}{\alpha^2}}}\,ds.$$

By the intermediate value theorem[8], for any small enough α we can find $\lambda_\alpha \in (\lambda-\varepsilon,\lambda+\varepsilon)$ such that

$$\int_0^1 \frac{1}{\sqrt{\frac{G(\alpha,\lambda+\varepsilon)-G(\alpha s,\lambda+\varepsilon)}{\alpha^2}}}\,ds = \frac{T_j}{\sqrt{2}}.$$

[8]Here we use without proof that $\lambda \to \int_0^1 \frac{1}{\sqrt{\frac{G(\alpha,\lambda)-G(\alpha s,\lambda)}{\alpha^2}}}$ is continuous for small $\alpha > 0$.

Then Theorem 2.3 implies that there is a solution with L^∞-norm α at this λ_α. One checks $\lambda_\alpha \to \lambda$ as $\alpha \to 0$, so $(0, \lambda)$ is a bifurcation point. □

Remark 2.6.

(a) In Corollary 2.5 we proved that the solutions of the boundary value problem bifurcate in $L^\infty(0,T) \times \mathbb{R}$ from the curve of trivial solutions $\{(0,\lambda) : \lambda \in \mathbb{R}\}$. One may check that this convergence not only holds in $L^\infty(0,T) \times \mathbb{R}$ but even in $C^k([0,T];\mathbb{R}) \times \mathbb{R}$ for all $k \in \mathbb{N}$. In other words, the solutions do not only converge uniformly to 0, but even all of their derivates converge uniformly.

(b) The condition "$\mu \mapsto g_z(0,\mu)$ is strictly monotone near λ" in (ii) may be checked via $g_{z\lambda}(0,\lambda) \neq 0$. We will come back to this condition in an abstract setting provided by the Crandall-Rabinowitz Theorem where this corresponds to the "Transversality Condition".

End of Lec03

3 Calculus in Banach spaces

In the following let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y), (Z, \|\cdot\|_Z)$ always denote real Banach spaces and $U \subset X$ is an open set with $x_0 \in U$. We denote by $\mathcal{L}(X;Z)$ the Banach space of bounded linear operators mapping X into Z. Its norm is given by

$$\|L\|_{\mathcal{L}(X;Z)} = \sup_{\|h\|_X \le 1} \|Lh\|_Z$$

The space of continuous n-linear maps $\mathcal{L}^n(X;Z)$ is a Banach space equipped with

$$\|L\|_{\mathcal{L}^n(X;Z)} = \sup_{\|h_1\|_X, \dots, \|h_n\|_X \le 1} \|L[h_1, \dots, h_n]\|_Z.$$

Definition 3.1. *A function $F : U \to Z$ is called ...*

(i) ... continuous at x_0 if for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|x - x_0\|_X < \delta \quad \Rightarrow \quad \|F(x) - F(x_0)\|_Z < \varepsilon.$$

In this case we write $F(x) = F(x_0) + o(1)$ as $x \to x_0$.

(ii) ... continuous in U if it is continuous at each $x \in U$.

(iii) ... Gâteaux-differentiable with respect to $h \in X$ at x_0 if

$$dF(x_0)[h] := \lim_{t \to 0} \frac{F(x_0 + th) - F(x_0)}{t} \quad \textit{exists.}$$

(iv) ... (Fréchet-)differentiable at x_0 if there is $L \in \mathcal{L}(X;Z)$ having the property that for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|x - x_0\|_X < \delta \quad \Rightarrow \quad \|F(x) - F(x_0) - L[x - x_0]\|_Z \le \varepsilon \|x - x_0\|_X.$$

In this case L is uniquely determined and we write $F'(x_0) := L$ and $F(x) = F(x_0) + F'(x_0)[x - x_0] = o(\|x - x_0\|_X)$ as $x \to x_0$.

(v) ... continuously (Fréchet-)differentiable in U, or $F \in C^1(U;Z)$, if the map $U \ni x \mapsto F'(x) \in \mathcal{L}(X;Z)$ is continuous.

(vi) ... twice differentiable at $x_0 \in U$ if the map $U \ni x \mapsto F'(x) \in \mathcal{L}(X;Z)$ is differentiable at x_0 according to (iv). We define $F''(x_0) \in \mathcal{L}^2(X;Z)$ by[9]

$$F''(x_0) : X \times X \to Z, \quad F''(x)[h_1, h_2] := ((F')'(x)[h_1])[h_2].$$

(vi) ... twice continuously differentiable, or $F \in C^2(U;Z)$, if the map $U \ni x \mapsto F''(x_0) \in \mathcal{L}^2(X;Z)$ is continuous.

[9]NB: For all $x \in U$ the function $(F')'(x)$ is a bounded linear operator from X to $\mathcal{L}(X;Z)$. As a consequence, for any given $h_1 \in X$, the function $(F')'(x)[h_1]$ is in $\mathcal{L}(X;Z)$. Applying this bounded linear operator to $h_2 \in X$ gives an element of Z, namely $((F')'(x)[h_1])[h_2]$.

Higher order derivative $F^{(k)}(x_0) \in \mathcal{L}^k(X;Z)$ are defined accordingly.

We shall also consider products of Banach spaces $X \times Y$ where the norm is given by $\|(x,y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$ (or an equivalent one).

Definition 3.2. *Assume $F : U \to Z$ where $U \subset X \times Y$ and $(x_0, y_0) \in U$. We call F differentiable with respect to Y at (x_0, y_0) if $F(x_0, \cdot) : Y \to Z$ is differentiable at y_0. In this case the partial derivative is an element of $\mathcal{L}(Y;Z)$ and is denoted by $F_y(x_0, y_0)$ or $\partial_y F(x_0, y_0)$.*

Short exercises:

- F (Fréchet-)differentiable at x_0 implies that F is Gâteaux-differentiable at x_0 with respect to all directions h, but not vice versa. Do we always have $F'(x)[h] = dF(x)[h]$ in this case?
- F (Fréchet-)differentiable at x_0 implies that F is continuous at x_0.
- F Gâteaux-differentiable at x_0 does not imply that F is continuous at x_0.
- Prove that $\mathcal{L}(X; \mathcal{L}^{n-1}(X;Z))$ and $\mathcal{L}^n(X;Z)$ are isomorphic for all $n \in \mathbb{N}$.

In order to use calculus in Banach spaces in an efficient way, we need a few computational rules. In the proofs one makes use of the well-known formulas from calculus in $\mathbb{R}$. This can be achieved by using

$$z_1 = z_2 \quad \Leftrightarrow \quad T[z_1 - z_2] = 0 \quad \forall T \in Z', \qquad \|z\|_Z = \sup_{\|T\|_{Z'} \le 1} |T[x]|$$

where $Z' = \mathcal{L}(Z;\mathbb{R})$ is the space of bounded linear functionals on Z. These facts, valid for any Banach space Z, rely on the Hahn-Banach Theorem.

Proposition 3.3.

(i) (Linearity) $F, G \in C^k(U;Z)$ implies $\alpha F + \beta G \in C^k(U;Z)$ for all $\alpha, \beta \in \mathbb{R}$ with

$$(\alpha F + \beta G)^{(k)} = \alpha F^{(k)} + \beta G^{(k)}.$$

(ii) (Chain rule) $F \in C^1(U;Y), G \in C^1(Y;Z)$ implies $G \circ F \in C^1(U;Z)$ with[10]

$$(G \circ F)'(x)[h] = G'(F(x))[F'(x)[h]] \qquad \forall h \in X.$$

[10] Alternative way of seeing this: $(G \circ F)'(x) = G'(F(x)) \circ F'(x) \in \mathcal{L}(Y;Z) \circ \mathcal{L}(X;Y) \subset \mathcal{L}(X;Z)$.

(iii) (Mean value theorem) $F \in C^1(U;Z)$ *and* $[x_1, x_2] \subset U$ *implies*

$$\|F(x_1) - F(x_2)\|_Z \leq \sup_{x \in [x_1, x_2]} \|F'(x)\|_{\mathcal{L}(X;Z)} \|x_1 - x_2\|.$$

(iv) (Schwarz' Theorem) If F *is twice differentiable at* x_0 *then*[11] $F''(x_0)[h_1, h_2] = F''(x_0)[h_2, h_1]$ *for all* $h_1, h_2 \in X$.

(v) (Taylor's Theorem) If $F \in C^k(U;Z)$ *then*

$$F(x) = F(x_0) + F'(x_0)[x - x_0] + \ldots + \frac{1}{k!} F^{(k)}(x_0)[x - x_0, \ldots, x - x_0] + o(\|x - x_0\|^k)$$

(vi) (Gâteaux vs. Fréchet) If $dF(x)[h]$ *exists for all* $x \in U, h \in X$ *and the map*

$$U \to \mathcal{L}(X;Z), \quad x \mapsto dF(x)$$

is continuous at x_0*, then* F *is Fréchet-differentiable at* x_0 *with* $F'(x_0) = dF(x_0)$.

Proof:
The proofs can be found in the book by Ambrosetti-Prodi [1]: More precisely,

(i) see [1, Proposition 1.4]
(ii) see [1, Proposition 1.4]
(iii) see [1, Theorem 1.8]
(iv) see Theorem 3.4 in [1]
(v) see [1, p.28-29]
(vi) see [1, Theorem 1.9]

We only prove (iii) here. Define $\hat{x}(t) := x_1 + t(x_2 - x_1)$. The chain rule (ii) implies $F \circ \hat{x} \in C^1([0,1];Z)$ with

$$\frac{d}{dt}\left(F(\hat{x}(t))\right) = F'(\hat{x}(t))[\hat{x}'(t)] = F'(x_1 + t(x_2 - x_1))[x_2 - x_1].$$

Applying the chain rule once more[12], we get for $T \in Z'$

$$\frac{d}{dt} T[F(\hat{x}(t))] = T\left[F'(x_1 + t(x_2 - x_1))[x_2 - x_1]\right].$$

We thus obtain

$$\begin{aligned}|T[F(x_2) - F(x_1)]| &= |(T \circ F \circ \hat{x})(1) - (T \circ F \circ \hat{x})(0)| \\ &= \left| \int_0^1 \frac{d}{dt}\left((T \circ F \circ \hat{x})(t)\right) dt \right|\end{aligned}$$

[11] This rule generalizes to higher order derivatives, see [1, Theorem 3.5].

[12] For any $T \in \mathcal{L}(Z;\mathbb{R})$ and $z \in Z$ we have $T'(z) = T$. In other words, the linear approximation of the map T at the point z is given by T itself.

$$
\begin{aligned}
&= \left| \int_0^1 T\left[F'(x_1 + t(x_2 - x_1))[x_2 - x_1]\right] dt \right| \\
&\le \int_0^1 \left|T\left[F'(x_1 + t(x_2 - x_1))[x_2 - x_1]\right]\right| dt \\
&\le \int_0^1 \|T\| \|F'(x_1 + t(x_2 - x_1))\| \|x_2 - x_1\| \, dt \\
&\le \|T\| \sup_{x \in [x_1, x_2]} \|F'(x)\| \cdot \|x_2 - x_1\|.
\end{aligned}
$$

Since this holds for all $T \in Z'$, we get

$$
\|F(x_2) - F(x_1)\| \le \sup_{x \in [x_1, x_2]} \|F'(x)\| \cdot \|x_2 - x_1\|.
$$

□

In applications, (vi) is often used to prove Fréchet-differentiability. This is advantageous because Gâteaux-derivatives are usually quite easy to compute.

Example 3.4.

(a) Let $X := C^2([0,1])$ and $Z := C([0,1])$, define $F : X \to Z$ via $F(u) := u'' + u^2$. We show

$$
F'(u_0)[h] = h'' + 2u_0 h \qquad (u_0, h \in X).
$$

Indeed, $Lh := h'' + 2u_0 h$ is a linear and bounded operator from X to Z because of

$$
\begin{aligned}
\|Lh\|_Z &= \|h''\|_{C([0,1])} + 2\|u_0 h\|_{C([0,1])} \\
&\le \|h\|_{C^2([0,1])} + 2\|u_0\|_{C([0,1])} \|h\|_{C([0,1])} \\
&\le (1 + 2\|u_0\|_{C([0,1])}) \|h\|_X.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\|F(u_0 + h) - F(u_0) - (h'' + 2u_0 h)\|_Z \\
&= \|((u_0 + h)'' + (u_0 + h)^2) - (u_0'' + u_0^2) - (h'' + 2u_0 h)\|_Z \\
&= \|h^2\|_Z = \|h\|_{C([0,1])}^2 \le \|h\|_X^2
\end{aligned}
$$

This proves $F'(u_0) = L$. Next we show $F''(u_0)[h_1, h_2] = 2h_1 h_2$. Indeed, we have

$$
\begin{aligned}
F'(u_0 + h_1)[h_2] - F'(u_0)[h_2] - 2h_1 h_2 &= \left(h_2'' + 2(u_0 + h_1)h_2\right) - \left(h_2'' + 2u_0 h_2\right) - 2h_1 h_2 \\
&= 0.
\end{aligned}
$$

(b) Let $X := L^p(\Omega)$, $Z := L^q(\Omega)$ and

$$
F : X \to Z, \quad u \mapsto \int_\Omega K(\cdot - y) u(y)^3 \, dy = K * (u^3 \mathbb{1}_\Omega)
$$

for some suitable measurable kernel function $K:\mathbb{R}^N\to\mathbb{R}^N$ and $\Omega\subset\mathbb{R}^N$.
Question: For which p,q do we have $F\in C^1(X;Y)$?
Young's convolution inequality implies

$$\|F(u)\|_Z \le \|K*(u^3\mathbb{1}_\Omega)\|_{L^q(\mathbb{R}^N)} \le \|K\|_{L^r(\mathbb{R}^N)}\|u^3\mathbb{1}_\Omega\|_{L^{p/3}(\mathbb{R}^N)} = \|K\|_{L^r(\mathbb{R}^N)}\|u\|^3_{L^p(\Omega)} < \infty$$

provided that $1\le p/3,q,r\le\infty$ and $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{p/3}$, i.e., $p\ge 3, \frac{1}{r}:=1-\frac{3}{p}+\frac{1}{q}\in[0,1]$.
We show that the same assumptions imply that F is differentiable on X with

$$F'(u)[h]=Lh:=3\int_\Omega K(\cdot-y)u(y)^2h(y)\,dy = 3K*(u^2h\mathbb{1}_\Omega).$$

Indeed, this defines a bounded linear operator (Prove it!) and

$$\begin{aligned}
\|F(u+h)-F(u)-Lh\|_Z &= \|(K*\mathbb{1}_\Omega(u+h)^3)-(K*\mathbb{1}_\Omega u^3)-3K*(u^2h\mathbb{1}_\Omega)\|_Z\\
&= \|K*\mathbb{1}_\Omega((u+h)^3-u^3-3u^2h)|_{L^q(\mathbb{R}^N)}\\
&\le \|K\|_{L^r(\mathbb{R}^N)}\|(u+h)^3-u^3-3u^2h\|_{L^{p/3}(\Omega)}\\
&= \|K\|_{L^r(\mathbb{R}^N)}\|3uh^2+h^3\|_{L^{p/3}(\Omega)}\\
&\le \|K\|_{L^r(\mathbb{R}^N)}(3\|uh^2\|_{L^{p/3}(\Omega)}+\|h^3\|_{L^{p/3}(\Omega)})\\
&\le \|K\|_{L^r(\mathbb{R}^N)}(3\|u\|_{L^p(\Omega)}\|h\|^2_{L^p(\Omega)}+\|h\|^3_{L^p(\Omega)})\\
&= O(\|h\|^2_{L^p(\Omega)}) = o(\|h\|_{L^p(\Omega)}).
\end{aligned}$$

Conclusion: If $p\in[3,\infty], q\in[1,\infty]$ and $K\in L^r(\mathbb{R}^N)$ for $\frac{1}{r}:=1-\frac{3}{p}+\frac{1}{q}\in[0,1]$, then F is Fréchet-differentiable and $F'(u)$ is given as above.
(Similar computations show that F is infinitely many times differentiable with $F''(u)[h_1,h_2]=6K*(uh_1h_2\mathbb{1}_\Omega),\ldots$)

End of Lec04

Theorem 3.5 (Implicit Function Theorem). *Let X,Y,Z be Banach spaces, $\Omega\subset X\times Y$ an open subset and $F:\Omega\to Z$ such that $F(x_0,y_0)=0$ for some $(x_0,y_0)\in\Omega$. Assume that*

(i) $x\mapsto F(x,y_0)$ is continuous at x_0,

(ii) $(x,y)\mapsto F_y(x,y)$ exists and is continuous as a map from Ω to $\mathcal{L}(Y;Z)$,

(iii) $F_y(x_0,y_0)\in\mathcal{L}(Y;Z)$ is invertible[13].

Then there is an open neighbourhood $U\times V\subset\Omega$ of (x_0,y_0) and precisely one function $\hat{y}:\overline{U}\to\overline{V}$ satisfying

$$(x,y)\in\overline{U}\times\overline{V},\ \ F(x,y)=0 \quad\Leftrightarrow\quad x\in\overline{U},\ y=\hat{y}(x).$$

Furthermore: If $F\in C^k(\Omega;Z)$, $k\in\mathbb{N}_0$, then $\hat{y}\in C^k(U;Y)$ and in case $k\ge1$ we have

$$\hat{y}'(x)[h]=-F_y(x,y)^{-1}[F_x(x,y)[h]]. \tag{3.1}$$

[13]This means that there is a bounded(!) linear operator $F_y(x_0,y_0)^{-1}\in\mathcal{L}(Z,;Y)$ such that $F_y(x_0,y_0)\circ F_y(x_0,y_0)^{-1}=\mathrm{id}_Y$ and $F_y(x_0,y_0)^{-1}\circ F(x_0,y_0)=\mathrm{id}_X$.

Proof:
W.l.o.g. assume $x_0 = 0$ and $y_0 = 0$; the general case may then be deduced from considering the map $\tilde{F}(x,y) := F(x_0{+}x, y_0{+}y)$ which is defined on $\tilde{\Omega} := \{(x,y) \in X{\times}Y : (x_0{+}x, y_0{+}y) \in \Omega\}$. We shall prove the claim for open balls

$$U := \{x \in X : \|x\| < \varepsilon\}, \qquad V := \{y \in Y : \|y\| < \rho\},$$

where $\rho > 0$ and then $\varepsilon \in (0,\rho)$ are chosen as follows:

$$\begin{aligned} \|F_y(0,0)^{-1}\|^{-1}_{\mathcal{L}(Z;Y)} \|F_y(x,y)^{-1}\|_{\mathcal{L}(Z;Y)} &\le 2 \quad \text{if } \|x\|_X, \|y\|_Y \le \rho \\ \|F_y(0,0)^{-1}\|_{\mathcal{L}(Z;Y)} \|F_y(x,y) - F_y(0,0)\|_{\mathcal{L}(Y;Z)} &\le \frac{1}{2} \quad \text{if } \|x\|_X, \|y\|_Y \le \rho \\ \|F_y(0,0)^{-1}\|_{\mathcal{L}(Z;Y)} \|F(x,0)\|_Z &\le \frac{\rho}{2} \quad \text{if } \|x\|_X \le \varepsilon. \end{aligned} \tag{3.2}$$

Such a choice is possible in view of (i),(ii),(iii) and it uses the fact that $(x,y) \mapsto F_y(x,y)^{-1}$ is well-defined and continuous at $(x_0,y_0) = (0,0)$, which in turn uses the Neumann series. To prove the first claim we choose an arbitrary $x \in \overline{U}$, i.e. $\|x\| \le \varepsilon$, and prove with the aid of Banach's Fixed Point Theorem that the map

$$T_x : \overline{V} \to Y,\ y \mapsto y - F_y(0,0)^{-1}[F(x,y)]$$

has a unique fixed point.

- **Selfmap:** For $\|y\|_Y \le \rho$ we have due to Proposition 3.3 (iii)

$$\begin{aligned} \|T_x(y)\|_Y &= \|F_y(0,0)^{-1}[F(x,y) - F_y(0,0)[y]]\|_Y \\ &\le \|F_y(0,0)^{-1}\|_{\mathcal{L}(Z;Y)} \left(\|F(x,y) - F(x,0) - F_y(0,0)[y]\|_Z + \|F(x,0)\|_Z\right) \\ &\le \|F_y(0,0)^{-1}\|_{\mathcal{L}(Z;Y)} \left(\sup_{t\in[0,1]} \|F_y(x,ty) - F_y(0,0)\| \cdot \|y\| + \|F(x,0)\|_Y \right) \\ &\overset{(3.2)}{\le} \frac{1}{2}\|y\|_Y + \frac{\rho}{2} \\ &\le \rho. \end{aligned}$$

  This proves $T_x(\overline{V}) \subset \overline{V}$.
- **Contraction:** For $\|y\|_Y, \|\tilde{y}\|_Y \le \rho$ we have again due to Proposition 3.3 (iii)

$$\begin{aligned} \|T_x(y) - T_x(\tilde{y})\|_Y &\le \|F_y(0,0)^{-1}\|_{\mathcal{L}(Z;Y)} \sup_{t\in[0,1]} \|F_y(x, \tilde{y} + t(y-\tilde{y})) - F_y(0,0)\|_{\mathcal{L}(Y;Z)} \cdot \|y - \tilde{y}\|_Y \\ &\overset{(3.2)}{\le} \frac{1}{2}\|y - \tilde{y}\|_Y. \end{aligned}$$

Since $(\overline{V}, d)$ is a nonempty complete metric space with respect to the metric $d(y_1,y_2) := \|y_1 - y_2\|_Y$ (being a nonempty closed subset of the Banach space Y), Banach's Fixed Point Theorem applies and proves the existence of precisely one $y =: \hat{y}(x) \in \overline{V}$ satisfying

$T_x(y) = y$, or equivalently $F(x,y) = 0$. We have thus proved the existence and uniqueness of a function $\hat{y} : \overline{U} \to \overline{V}$ satisfying $F(x,\hat{y}(x)) = 0$ for all $x \in \overline{U}$.

Continuity of $\hat{y}$, $k = 0$: We show that if F is continuous then $\hat{y}$ is continuous as well. Indeed, for $x \in U$ and $\|h\|_X$ sufficiently small we have $x + h \in U$ and hence T_{x+h} has Lipschitz constant $\le \frac{1}{2}$ by our computations above. This implies

$$\begin{aligned}
&\|\hat{y}(x+h) - \hat{y}(x)\|_Y \\
&= \|T_{x+h}(\hat{y}(x+h)) - T_x(\hat{y}(x))\|_Y \\
&\le \|T_{x+h}(\hat{y}(x+h)) - T_{x+h}(\hat{y}(x))\|_Y + \|T_{x+h}(\hat{y}(x)) - T_x(\hat{y}(x))\|_Y \\
&\le \frac{1}{2}\|\hat{y}(x+h) - \hat{y}(x)\|_Y + \|F_y(0,0)^{-1}[F(x+h,\hat{y}(x)) - F(x,\hat{y}(x)]\|_Y \\
&\le \frac{1}{2}\|\hat{y}(x+h) - \hat{y}(x)\|_Y + \|F_y(0,0)^{-1}\|_{\mathcal{L}(Z;Y)}\|F(x+h,\hat{y}(x)) - F(x,\hat{y}(x))\|_Z
\end{aligned} \tag{3.3}$$

Since $F(\cdot,\hat{y}(x))$ is continuous at x by assumption, we conclude $\hat{y}(x+h) \to \hat{y}(x)$ as $\|h\|_X \to 0$, so $\hat{y}$ is continuous at x. Since $x \in U$ was arbitrary, we conclude $\hat{y} \in C(U;Y)$.

Differentiability of $\hat{y}$, $k = 1$: We now assume $F \in C^1(\Omega;Z)$ and our aim is to show $\hat{y} \in C^1(U;V)$ as well as

$$\hat{y}'(x)[h] = -F_y(x,\hat{y}(x))^{-1}[F_x(x,\hat{y}(x))[h]].$$

Since F is differentiable at x, (3.3) implies $\|\hat{y}(x+h) - \hat{y}(x)\|_Y \le C\|h\|_X$ for some $C > 0$ and small $\|h\|_X$. Hence, for any given $\varepsilon > 0$ we can choose $\delta > 0$ such that $\|h\|_X \le \delta$ implies

$$\begin{aligned}
&\|F(x+h,\hat{y}(x+h)) - F(x,\hat{y}(x)) - F'(x,\hat{y}(x))[(h,\hat{y}(x+h) - \hat{y}(x)]\|_Z \\
&\le \varepsilon^*\|(h,\hat{y}(x+h) - \hat{y}(x))\|_{X\times Y} \\
&= \varepsilon^*(\|h\|_X + \|\hat{y}(x+h) - \hat{y}(x)\|_Y) \\
&\le (1+C)\varepsilon^*\|h\|_X
\end{aligned}$$

where $\varepsilon^* := \frac{\varepsilon}{2(1+C)}\|F_y(0,0)^{-1}\|_{\mathcal{L}(Z;Y)}^{-1}$. On the other hand,

$$\begin{aligned}
&F(x+h,\hat{y}(x+h)) - F(x,\hat{y}(x)) - F'(x,\hat{y}(x))[(h,\hat{y}(x+h) - \hat{y}(x)] \\
&= -F_x(x,\hat{y}(x))[h] - F_y(x,\hat{y}(x))[\hat{y}(x+h) - \hat{y}(x)] \\
&= -F_y(x,\hat{y}(x))\Big[\hat{y}(x+h) - \hat{y}(x) + F_y(x,\hat{y}(x))^{-1}[F_x(x,\hat{y}(x))[h]]\Big],
\end{aligned}$$

which implies

$$\begin{aligned}
&\|\hat{y}(x+h) - \hat{y}(x) + F_y(x,\hat{y}(x))^{-1}[F_x(x,\hat{y}(x))[h]]\|_Y \\
&\le \|F_y(x,\hat{y}(x))^{-1}\|_{\mathcal{L}(Z;Y)}(1+C)\varepsilon^*\|h\|_X
\end{aligned}$$

$$\le 2(1+C)\varepsilon^* \|F_y(0,0)^{-1}\|_{\mathcal{L}(Z;Y)} \|h\|_X$$
$$\le \varepsilon \|h\|_X.$$

This proves the claim.

Higher differentiability of $\hat{y}$, $k \ge 2$: Exploiting the formula

$$\hat{y}'(x)[h] = -F_y(x, \hat{y}(x))^{-1}[F_x(x, \hat{y}(x))[h]]$$

and the chain rule Proposition 3.3 (ii) we find that $F \in C^k(\Omega; Z)$ implies $\hat{y} \in C^k(U; Y)$. □

End of Lec05

A variant of the Implicit Function Theorem is the Local Inversion Theorem.

Theorem 3.6 (Local Inversion Theorem). *Let Y, Z be Banach spaces, $\Omega \subset Y$ an open subset with $y_0 \in \Omega$, $G \in C^1(\Omega; Z)$ such that $G'(y_0)$ is invertible. Then there are open neighbourhoods V, W of $y_0, G(y_0)$ such that $G : \bar{V} \to \bar{W}$ admits a continuously differentiable inverse $G^{-1} \in C^1(W; Y)$ satisfying*

$$(G^{-1})'(w) = G'(G^{-1}(w))^{-1} \quad \text{for all } w \in W.$$

Proof:
Set $z_0 := G(y_0) \in Z$. We apply the IFT (Theorem 3.5) to the function $F : Z \times \Omega, (z, y) \mapsto G(y) - z$. This function is continuously differentiable on $Z \times \Omega$ with $F_y(z_0, y_0) = G'(y_0)$, which is invertible. The IFT yields open neighbourhoods W, V of z_0, y_0 and $\hat{y} \in C^1(W; Y)$ such that $F(z, y) = 0$ and $(z, y) \in W \times V$ is equivalent to $z \in V, y = \hat{y}(z)$. So $G^{-1} := \hat{y}$ gives the claim. □

We now show that these abstract results can be used to prove the existence of solutions to nonlinear elliptic boundary value problems. We start with an application of the Local Inversion Theorem to

$$-\Delta u = f(x, u) + \varepsilon g(x), \qquad u \in H_0^1(\Omega) \tag{3.4}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$. We recall that $H_0^1(\Omega)$ is the completion of the test functions $C_c^\infty(\Omega)$ with respect to the norm $u \mapsto \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$ or, equivalently by Poincaré's inequality, with respect to the norm $u \mapsto \|\nabla u\|_{L^2(\Omega)}$. We shall need the bounded linear operator

$$(-\Delta)^{-1} : L^{\frac{2N}{N+2}}(\Omega) \to H_0^1(\Omega), \qquad f \mapsto u$$

where $u \in H_0^1(\Omega)$ denotes the unique weak solution of $-\Delta u = f$ in Ω.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $g \in L^2(\Omega)$ and $f \in C^1(\overline{\Omega} \times \mathbb{R})$ be given such that*

(i) $|f_z(x,z)| \le C(1+|z|^{p-1})$ *for all $x \in \overline{\Omega}, z \in \mathbb{R}$ where the exponent p satisfies*

$$1 < p < \infty \quad \text{in case } N \in \{1,2\}, \qquad 1 < p \le \frac{N+2}{N-2} \quad \text{in case } N \ge 3.$$

(ii) $f(x,0) = f_z(x,0) = 0$ *for all $x \in \overline{\Omega}$.*

Then there is a (small) $\varepsilon_0 > 0$ such that (3.4) has a (unique small) solution $u_\varepsilon \in H_0^1(\Omega)$ for any given $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Moreover, $\varepsilon \mapsto u_\varepsilon$ is continuously differentiable.

Proof:
We apply the Local Inversion Theorem to the map

$$G : H_0^1(\Omega) \to H_0^1(\Omega), \quad u \mapsto u - (-\Delta)^{-1}(f(\cdot,u))$$

at the point $u_0 = 0$ in order to solve the equation $G(u) = (-\Delta)^{-1}(\varepsilon g)$, which is equivalent to (3.4). The function G is continuously differentiable[14] and we know $G(0) = 0$ thanks to $f(x,0) = 0$. Furthermore, $f_z(x,0) = 0$ for all $x \in \Omega$ implies

$$G'(0)[h] = h - (-\Delta)^{-1}(\underbrace{f_z(\cdot,0)}_{\equiv 0} h) = h.$$

So $G'(0) = I : H_0^1(\Omega) \to H_0^1(\Omega)$ is invertible. The Local Inversion Theorem provides open neighbourhoods U, V of the origin such that $G : U \to V$ admits a continuously differentiable inverse $G^{-1} \in C^1(V;U)$. Now choose $\varepsilon_0 > 0$ such that $(-\Delta)^{-1}(\varepsilon g) \in V$ for all $|\varepsilon| < \varepsilon_0$. For such ε we have a unique $u_\varepsilon \in U$ satisfying $G(u_\varepsilon) = (-\Delta)^{-1}(\varepsilon g)$, namely

$$u_\varepsilon := G^{-1}\left((-\Delta)^{-1}(\varepsilon g)\right).$$

[14] This is a nontrivial fact and relies on the existence of a continuous embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ where $q < \infty$ if $N = 1,2$ and $q \le \frac{2N}{N-2}$ if $N \ge 3$. This is Sobolev's Embedding Theorem. The main aspects of the nontrivial proof can be found in [1, Theorem 2.9]. We only show why $1 < p \le \frac{N+2}{N-2}$ is needed in the case $N \ge 3$. We want to show that for any given $u \in H_0^1(\Omega)$

$$G'(u)[h] = h - (-\Delta)^{-1}(f_z(\cdot,u)h)$$

defines a bounded linear operator $H_0^1(\Omega) \to H_0^1(\Omega)$. Indeed, $u, h \in H_0^1(\Omega)$ implies $u, h \in L^{\frac{2N}{N-2}}(\Omega)$ and thus for some $C_1, \dots, C_4 > 0$

$$\begin{aligned}
\|f_z(\cdot,u)h\|_{\frac{2N}{N+2}} &\le C\|(1+|u|)^{p-1}h\|_{\frac{2N}{N+2}} \\
&\le C_1(\|1\cdot h\|_{\frac{2n}{N+2}} + \|1\cdot |u|^{p-1}h\|_{\frac{2N}{N+2}}) \\
&\le C_2(\|1\|_N\|h\|_2 + \|1\|_{\frac{2N}{N+2-p(N-2)}}\||u|^{p-1}\|_{\frac{2N}{(N-2)(p-1)}}\|h\|_{\frac{2N}{N-2}}) \\
&\le C_3(\|h\|_2 + \|u\|_{\frac{2N}{N-2}}^{p-1} + \|h\|_{\frac{2N}{N-2}}) \\
&\le C_4(\|h\|_2 + \|u\|_{H_0^1(\Omega)}^{p-1} + \|h\|_{H_0^1(\Omega)})
\end{aligned}$$

Here we used Hölder's inequality and the boundedness of Ω.

This u_ε is a weak solution of (3.4) and $\varepsilon \mapsto u_\varepsilon$ is continuously differentiable. □

Note that the result can equally be proved with the IFT, which is applied to the map

$$H_0^1(\Omega) \times \mathbb{R} \to H_0^1(\Omega), \quad (u,\varepsilon) \mapsto u - (-\Delta)^{-1}(f(\cdot,u) + \varepsilon g(\cdot)).$$

Now we consider the IFT in the context of nonlinear boundary value problem for ordinary differential equations:

$$-u''(x) = f(x,u,u',\lambda) \text{ in } (0,1), \qquad u(0) = u(1) = 0. \tag{3.5}$$

Theorem 3.8. *Let $f \in C^1([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ satisfy $f(x,0,0,\lambda_0) = 0$ and assume that the linearized problem at $(0,\lambda_0)$, namely*[15]

$$-\phi''(x) = f_u(x,0,0,\lambda_0)\phi(x) + f_p(x,0,0,\lambda_0)\phi'(x) \text{ in } (0,1), \qquad \phi(0) = \phi(1) = 0,$$

only admits the trivial solution. Then there is a continuously differentiable curve $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \ni \lambda \mapsto u_\lambda \in C^2([0,1])$ such that $u_0 = 0$ and (u_λ, λ) is a solution[16] *of* (3.5).

Proof:
We apply the IFT to $X = \mathbb{R}, Y = \{u \in C^2([0,1]) : u(0) = u(1) = 0\}, Z = C([0,1])$ where

$$F : X \times Y \to Z, \ (\lambda, u) \mapsto u'' + f(\cdot,u,u',\lambda).$$

Then $F \in C^1(X \times Y, Z)$ with $F(\lambda_0, 0) = 0$ and

$$F_u(\lambda,u)[h] = h'' + f_u(\cdot,u,u',\lambda)h + f_p(\cdot,u,u',\lambda)h'.$$

By assumption the operator $F_u(\lambda_0,0)$ is injective. To see that it is surjective as well consider the equation $F_u(\lambda_0,0)[h] = z$ for $h \in Y, z \in Z$, which is equivalent to

$$(I-K)h := h - (-\Delta)^{-1}(f_u(\cdot,0,0,\lambda_0)h + f_p(\cdot,0,0,\lambda_0)h') = (-\Delta)^{-1}(z).$$

Here, $(-\Delta)^{-1} : Z \to Y$ is again the bounded linear operator[17] that maps $z \in Z$ onto the unique solution $\phi \in Y$ of $-\Delta\phi = z$. Given that $Y \ni h \mapsto f_u(\cdot,0,0,\lambda_0)h + f_p(\cdot,0,0,\lambda_0)h' \in Z$ is compact, we have that $K : Y \to Y$ is a bounded linear and compact operator. So Fredholm's Alternative [6, Theorem 5(iv)] implies that $I - K$ is invertible (being injective), so $h = (I-K)^{-1}((-\Delta)^{-1}z)$. This proves surjectivity of $F_u(\lambda_0,0)$. We thus conclude that $F_u(\lambda_0,0)$ is an invertible bounded linear operator. Hence, the IFT applies and gives the result. □

End of Lec 06

[15] f_u, f_p denote the derivatives with respect to u, u', respectively.
[16] If $f(x,0,0,\lambda) = 0$ for all $x \in \Omega$ and λ close to λ_0, then these solutions u_λ are trivial, i.e., $u_\lambda = 0$.
[17] The existence of such an operator requires some work! This is the content of a course on "Boundary and Eigenvalue problems". In the present ODE situation, the matter is easier because one can show

$$(-\Delta)^{-1}h(x) = \int_0^1 G(x,t)h(t)\,dt, \qquad \text{where} \quad G(x,t) = \begin{cases} t(1-x) & 0 \le t \le x, \\ x(1-t) & ,x \le t \le 1. \end{cases}$$

These two results show how the Local Inversion Theorem and the Implicit Function Theorem may be applied to prove the existence of solutions to nonlinear elliptic boundary value problems. We now show that bifurcation can only occur when the IFT is not applicable.

Proposition 3.9. *Let X, Z be Banach spaces, $F \in C^1(\mathbb{R} \times X, Z)$ with $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. If λ_0 is a bifurcation point, then $F_x(\lambda_0, 0)$ is not invertible, i.e. $0 \in \sigma(F_x(\lambda_0))$.*

Proof:
We assume that $F_x(\lambda_0, 0)$ is invertible. Then the IFT shows that all solutions (λ, x) close to $(\lambda_0, 0)$ are of the form $x = \hat{x}(\lambda)$ for some continuously differentiable curve $\hat{x} : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$. In view of $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$ we conclude $\hat{x}(\lambda) = 0$ whenever $|\lambda - \lambda_0| < \varepsilon$. Hence, all solutions in a small neighbourhood are trivial, so $(\lambda_0, 0)$ is not a bifurcation point. □

Remark 3.10. Let $F \in C^1(\mathbb{R} \times X; Z)$ with $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

(a) Very often, the function F is given by $F(x, \lambda) = Lx - \lambda x - G(x, \lambda)$ where L is some bounded linear operator and $G \in C^1(X \times \mathbb{R})$ satisfies $G_x(0, \lambda_0) = 0$. Then Proposition 3.9 states that bifurcation can only occur at $\lambda = \lambda_0$ if λ_0 is an eigenvalue of L. For instance, $\{\frac{k^2\pi^2}{T^2} : k \in \mathbb{N}\}$ is in fact the spectrum of the linearized operator associated with (2.1), namely $\theta \mapsto -\theta''$ with homogeneous Dirichlet boundary conditions $\theta(0) = \theta(T) = 0$. More precisely, $\frac{T^2}{k^2\pi^2}$ are (all) the eigenvalues of the compact selfadjoint operator $\iota \circ (-\Delta)^{-1} : L^2(\Omega) \to H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

(b) If bifurcation occurs at $(\lambda_0, 0)$ then the corresponding solutions look like the eigenfunction close to the bifurcation point. Indeed, assume $(\lambda_n, x_n) \to (\lambda_0, 0)$ and $x_n \|x_n\|_X^{-1} \to \phi$. Then, if F is continuously differentiable,

$$\begin{aligned} 0 &= \|x_n\|^{-1}(F(\lambda_n, x_n) - F(\lambda_n, 0) - F_x(\lambda_n, 0)[x_n] + F_x(\lambda_n, 0)[x_n]) \\ &= \|x_n\|^{-1}\left(F_x(\lambda_n, 0)[x_n] + O\left(\sup_{t\in[0,1]} \|F_x(\lambda_n, tx_n) - F_x(\lambda_n, 0)\|_{\mathcal{L}(X;Z)}\|x_n\|_X\right)\right) \\ &= F_x(\lambda_n, 0)\left[\frac{x_n}{\|x_n\|_X}\right] + o(1) \\ &= F_x(\lambda_0, 0)[\phi] + o(1). \end{aligned}$$

As a consequence, the "direction of bifurcation" with respect to X is given by the eigenfunctions. We shall make this more precise on the following section. Recall that the functions $t \mapsto \sin(k\pi t/T)$ showing up in Corollary 2.5 are indeed eigenfunctions of $\theta \mapsto -\theta''$ with homogeneous Dirichlet boundary conditions (eigenvalue $= \frac{\pi^2 k^2}{T^2}$).

4 Bifurcation from a simple eigenvalue

In the following let always X, Z denote real Banach spaces and assume that $F \in C^1(\mathbb{R} \times X; Z)$ satisfies $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. In other words, there is a trivial solution family. Proposition 3.9 tells us that the only candidates for bifurcation points are given by values λ_0 such that $F_x(\lambda_0, 0)$ is not invertible. So it is a natural question to ask which of these values is indeed a bifurcation point. In this section we discuss the easiest case when λ_0 is a simple eigenvalue. The corresponding bifurcation result - the Crandall-Rabinowitz Theorem [4] Theorem 4.4 - is often referred to as "bifurcation from a simple eigenvalue". We impose the following assumption:

(S) $F_x(\lambda_0, 0) : X \to Z$ is a $(1,1)$-Fredholm operator.

Definition 4.1. *Let $p, q \in \mathbb{N}_0$. A linear operator $L \in \mathcal{L}(X; Z)$ is called (p,q)-Fredholm if $\dim(\ker(L)) = p$ and $\operatorname{codim}(\operatorname{ran}(L)) := \dim(Z/\operatorname{ran}(L)) = q$. Its Fredholm index is $p - q \in \mathbb{Z}$.*

We recall:

$$\ker(L) = \{x \in X : Lx = 0\},$$
$$\operatorname{ran}(L) = \{z \in Z : \exists x \in X \; Lx = z\}.$$

It is known [2, pp.37-38] that finite-dimensional and finite-codimensional subspaces of Banach spaces are complemented. In particular, if L is a (p,q)-Fredholm operator then there are closed subspaces $\tilde{X} \subset X$ and $\tilde{Z} \subset Z$ such that

$$X = \ker(L) \oplus \tilde{X}, \qquad Z = \operatorname{ran}(L) \oplus \tilde{Z}. \tag{4.1}$$

If X (resp. Z) has infinite dimension then $\tilde{X}$ (resp. $\operatorname{ran}(L)$) is infinite-dimensional, whereas $\dim(\ker(L)) = p, \dim(\tilde{Z}) = q$. When (S) holds we have $p = q = 1$ and we can make use of the following auxiliary result.

Proposition 4.2. *Assume that $L \in \mathcal{L}(X; Z)$ is $(1,1)$-Fredholm*[18]*. Then there is a bounded linear functional $\psi \in Z' \setminus \{0\}$ such that*

$$z \in \operatorname{ran}(L) \quad \Leftrightarrow \quad \psi(z) = 0.$$

In other words, $\operatorname{ran}(L) = \ker(\psi)$.

For a proof see [2, p.38, Example 1]. Besides the simplicity condition (S) we will need the "Transversality Condition"

[18] Actually, for this result to hold true we only need that $\operatorname{ran}(L)$ has codimension 1.

(T) $F_{x\lambda}(\lambda_0,0)[\phi] \notin \operatorname{ran}(L)$ where $\ker(L) = \mathbb{R}\phi$.

Note that (T) does not hold if $F(\lambda,x) = F_1((\lambda-\lambda_0)^2,x)$ or $F(\lambda,x) = F_2(\lambda_0,x)$ for some $F_1,F_2 : \mathbb{R}\times X \to Z$. We remark that here and in the following we identify[19]

$$F_{x\lambda}(\lambda_0,0)[\phi] \simeq F_{x\lambda}(\lambda_0,0)[\phi,1] \in Z.$$

The first step is the Lyapunov-Schmidt reduction in order to reduce the infinite-dimensional problem $F(\lambda,x) = 0$ to a finite-dimensional one close to $(\lambda_0,0)$.

End of Lec07

Lemma 4.3 (Lyapunov-Schmidt reduction). *Assume $F \in C^1(\mathbb{R}\times X; Z)$ satisfies (S) and $F(\lambda,0) = 0$ for all $\lambda \in \mathbb{R}$. Then there is a uniquely determined $\hat{y} \in C^1((\lambda_0-\varepsilon,\lambda_0+\varepsilon)\times(-\varepsilon,\varepsilon);\tilde{X})$ and an open neighbourhood U of $(\lambda_0,0) \subset \mathbb{R}\times X$ such that*

$$F(\lambda,x) = 0 \quad \text{for } (\lambda,x)\in U \quad \Leftrightarrow \quad x = \hat{y}(\lambda,s) + s\phi \quad \text{and}$$
$$\psi(F(\lambda,\hat{y}(\lambda,s)+s\phi)) = 0 \quad \text{for } |\lambda-\lambda_0|,|s| < \varepsilon.$$

We have $\hat{y}(\lambda_0,0) = \hat{y}_\lambda(\lambda_0,0) = \hat{y}_s(\lambda_0,0) = 0$ and if F is k times continuouly differentiable for $k\in\mathbb{N}$, then so is $\hat{y}$.

Proof:
We want to apply the Implicit Function Theorem. In view of (4.1) we make the ansatz $x = y + s\phi$ with $y \in \tilde{X}$ so that $F(\lambda,x) = 0$ is equivalent to $F(\lambda,y+s\phi) = 0$. Choose $\phi^* \in \tilde{Z}$ such that $\psi(\phi^*) = 1$ and define the projector $\Pi : Z \to \operatorname{ran}(Z)$, $z \mapsto z - \psi(z)\phi^*$. In order to solve the simpler equation $\Pi(F(\lambda,y+s\phi)) = 0$ with the aid of the IFT we define the continuously differentiable function

$$G : \mathbb{R}\times\mathbb{R}\times\tilde{X} \to \operatorname{ran}(F_x(\lambda_0,0)), \quad (\lambda,s,y) \mapsto \Pi(F(\lambda,y+s\phi)).$$

We have $G(\lambda_0,0,0) = 0$ and

$$G_y(\lambda_0,0,0) = F_x(\lambda_0,0)[\cdot] : \tilde{X} \to \operatorname{ran}(F_x(\lambda_0,0))$$

is invertible. The Implicit Function Theorem shows that there is an open neighbourhood $(\lambda_0-\varepsilon,\lambda_0+\varepsilon)\times(-\varepsilon,\varepsilon)\times W$ in $\mathbb{R}\times\mathbb{R}\times\tilde{X}$ and hence an open neighbourhood $U := \{(\lambda,y+s\phi) : |\lambda-\lambda_0| < \varepsilon, |s| < \varepsilon, y \in W\} \subset \mathbb{R}\times X$ of $(\lambda_0,0)$ and a uniquely determined function $\hat{y}$ such that $G(\lambda,s,\hat{y}(\lambda,s)) = 0$ as well as $\hat{y}(\lambda_0,0) = 0$. The formula for the derivatives of the implicit function (3.1) and

$$G_\lambda(\lambda_0,0,0) = \Pi(F_\lambda(\lambda_0,0)) = \Pi(0) = 0,$$
$$G_s(\lambda_0,0,0) = \Pi(F_x(\lambda_0,0)[\phi]) = 0$$

imply $\hat{y}_\lambda(\lambda_0,0) = \hat{y}_s(\lambda_0,0) = 0$. Hence,

$$F(\lambda,\hat{y}(\lambda,s)+s\phi) = \Pi(F(\lambda,\hat{y}(\lambda,s)+s\phi)) + \psi(F(\lambda,\hat{y}(\lambda,s)+s\phi))\cdot\phi^*$$

[19]Strictly speaking $F_{x\lambda}(\lambda_0,0) \in \mathcal{L}^2(X\times\mathbb{R};Z)$.

$$= G(\lambda, s, \hat{y}(\lambda, s)) + \psi(F(\lambda, \hat{y}(\lambda, s) + s\phi)) \cdot \phi^*$$
$$= \psi(F(\lambda, \hat{y}(\lambda, s) + s\phi)) \cdot \phi^*.$$

□

So we are left to find zeros of the function

$$\mathfrak{F} : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \times (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad (\lambda, s) \mapsto \psi(F(\lambda, \hat{y}(\lambda, s) + s\phi)).$$

To detect bifurcation from the trivial solution family the task is to construct nontrivial zeros (λ, s) that are characterized by $s \neq 0$. In fact, if $s \neq 0$ and $|s|$ is small, then $x = \hat{y}(\lambda, s) + s\phi$ is non-zero as well.

We shall exploit that $\mathfrak{F}$ has the same regularity as F, so is at least twice continuously differentiable. Later we will need some derivatives[20] of $\mathfrak{F}$ at $(\lambda_0, 0)$:

$$\begin{aligned} \mathfrak{F}_\lambda(\lambda_0, 0) &= 0, \\ \mathfrak{F}_s(\lambda_0, 0) &= 0, \\ \mathfrak{F}_{ss}(\lambda_0, 0) &= \psi(F_{xx}(\lambda_0, 0)[\phi, \phi]), \\ \mathfrak{F}_{s\lambda}(\lambda_0, 0) &= \psi(F_{x\lambda}(\lambda_0, 0)[\phi]), \\ \mathfrak{F}_{sss}(\lambda_0, 0) &= \psi\Big(F_{xxx}(\lambda_0, 0)[\phi, \phi, \phi] - 3F_{xx}(\lambda_0, 0)[\zeta, \phi]\Big). \end{aligned} \tag{4.2}$$

Here, one uses $\hat{y}_{ss}(\lambda_0, 0) = -\zeta$ where $\zeta \in \tilde{X}$ is uniquely determined by $F_x(\lambda_0, 0)[\zeta] = F_{xx}(\lambda_0, 0)[\phi, \phi]$.

[20] The formulas (4.2) are a consequence of specializing the following formulas to $(\lambda, s) = (\lambda_0, 0)$ and exploiting $\hat{y}(\lambda_0, 0) = \hat{y}_\lambda(\lambda_0, 0) = \hat{y}_s(\lambda_0, 0) = 0$ as well as $\psi|_{\mathrm{ran}(F_x(\lambda_0,0))} \equiv 0$:

$$\begin{aligned} \mathfrak{F}_\lambda(\lambda, s) &= \psi\Big(F_\lambda(\lambda, \hat{y}(\lambda, s) + s\phi) + F_x(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_\lambda(\lambda, s)]\Big), \\ \mathfrak{F}_s(\lambda, s) &= \psi(F_x(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_s(\lambda, s) + \phi]), \\ \mathfrak{F}_{ss}(\lambda, s) &= \psi\Big(F_{xx}(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_s(\lambda, s) + \phi, \hat{y}_s(\lambda, s) + \phi] + F_x(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_{ss}(\lambda, s)]\Big), \\ \mathfrak{F}_{s\lambda}(\lambda, s) &= \psi\Big(F_{x\lambda}(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_s(\lambda, s) + \phi] + F_{xx}(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_s(\lambda, s), \hat{y}_\lambda(\lambda, s)] \\ &\qquad + F_x(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_{s\lambda}(\lambda, s)]\Big), \\ \mathfrak{F}_{sss}(\lambda, s) &= \psi\Big(F_{xxx}(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_s(\lambda, s) + \phi, \hat{y}_s(\lambda, s) + \phi, \hat{y}_s(\lambda, s) + \phi] \\ &\qquad + 3F_{xx}(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_{ss}(\lambda, s), \hat{y}_s(\lambda, s) + \phi] \\ &\qquad + F_x(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_{sss}(\lambda, s)]\Big). \end{aligned}$$

Theorem 4.4 (Crandall-Rabinowitz, 1971). *Assume $F \in C^2(\mathbb{R}\times X; Z)$ satisfies $F(\lambda,0) = 0$ for all $\lambda \in \mathbb{R}$ as well as (S),(T). Then there is an $\varepsilon > 0$ and a curve $(\hat\lambda, \hat x) \in C^1((-\varepsilon,\varepsilon); \mathbb{R}\times X)$ such that $\hat x(0) = 0, \hat\lambda(0) = \lambda_0, \hat x'(0) = \phi$ and*

$$F(\hat\lambda(s), \hat x(s)) = 0 \qquad \forall s \in (-\varepsilon, \varepsilon).$$

Moreover, in some open neighbourhood of $(\lambda_0, 0)$ all nontrivial solutions of $F(\lambda, x) = 0$ lie on this curve.

Proof:
By the previous lemma it suffices to find nontrivial zeros (λ, s) of $\mathfrak{F}$ in a neighbourhood of $(\lambda_0, 0)$. In view of $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$ we have $\mathfrak{F}(\lambda, 0) = 0$. Define

$$\Phi : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \times (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad (\lambda, s) \mapsto \int_0^1 \mathfrak{F}_s(\lambda, ts)\, dt \; = \begin{cases} \frac{\mathfrak{F}(\lambda,s)}{s} & , s \neq 0 \\ \mathfrak{F}_s(\lambda, 0) & , s = 0. \end{cases}$$

Since $\mathfrak{F}$ is twice continuously differentiable near $(\lambda_0, 0)$, Φ is continuously differentiable near $(\lambda_0, 0)$. Moreover, the formulas (4.2) and the Transversality Condition (T) imply

$$\Phi(\lambda_0, 0) = \mathfrak{F}_s(\lambda_0, 0) = 0, \qquad \Phi_\lambda(\lambda_0, 0) = \mathfrak{F}_{\lambda s}(\lambda_0, 0) \overset{(T)}{\neq} 0.$$

So the IFT provides $\varepsilon > 0$ and a uniquely determined function $\hat\lambda \in C^1((-\varepsilon, \varepsilon); \mathbb{R})$ such that $\Phi(\hat\lambda(s), s) = 0$ with $\hat\lambda(0) = \lambda_0$. This implies, for $s \neq 0$, $\mathfrak{F}(\hat\lambda(s), s) = 0$. The previous Lemma then yields

$$F(\hat\lambda(s), \hat x(s)) = 0 \quad (0 < |s| < \varepsilon)$$

where $\hat x(s) := \hat y(\hat\lambda(s), s) + s\phi$ satisfies $\hat x(0) = 0, \hat x'(0) = \phi$. since all nontrivial zeros of F close to $(\lambda_0, 0)$ are zeros of Φ, we obtain that all nontrivial zeros close to $(\lambda_0, 0)$ are given by $(\lambda, x) = (\hat\lambda(s), \hat x(s))$ for some $s \neq 0$. □

End of Lec08

Remark 4.5.

(a) The solutions obtained by the Crandall-Rabinowitz Theorem are nontrivial for small enough $|s| > 0$. This follows from $\hat x'(0) = \phi \neq 0$, so $\hat x(s) = s\phi + o(s)$ as $s \to 0$. The latter also shows that the solutions bifurcate in the direction of the eigenfunction, which we have already observed in the context of the pendulum equation in Corollary 2.5 (i), see Remark 3.10. This means that the solutions "look like" the eigenfunction whenever s is zero. More precisely,

$$\lim_{s \to 0} \frac{\hat x(s)}{s} = \phi$$

(b) Under the symmetry assumption $F(\lambda, -x) = -F(\lambda, x)$ for all $x \in X, \lambda \in \mathbb{R}$ we get

$$\hat x(-s) = -\hat x(s), \qquad \hat\lambda(-s) = \hat\lambda(s) \qquad (|s| < \varepsilon).$$

Indeed, the proof of Lemma 4.3 shows $\Pi F(\lambda, x) = 0$ if and only if $x = \hat{y}(\lambda, s) + s\phi$. For any given zero (λ, x) of that equation, $(\lambda, -x)$ is again a solution. So $x = \hat{y}(\lambda, s) + s\phi$ implies $-x = -\hat{y}(\lambda, s) - s\phi$. On the other hand, $(\lambda, -x)$ is a zero and thus $-x = \hat{y}(\lambda, t) + t\phi$ for some $t \in \mathbb{R}$. We infer

$$-\hat{y}(\lambda, s) - s\phi = \hat{y}(\lambda, t) + t\phi,$$

hence $t = -s$ and $\hat{y}(\lambda, -s) = -\hat{y}(\lambda, s)$. This proves $\hat{x}(s) = -\hat{x}(-s)$ because of $\hat{x}(s) = \hat{y}(\lambda, s) + s\phi$. In a similar way, $\hat{\lambda}(s) = \hat{\lambda}(-s)$ follows from the proof of Theorem 4.4.

(c) If $F \in C^{k+1}(X \times \mathbb{R}; Z)$ then $\mathfrak{F}$ is C^{k+1}, Φ is C^k and hence $\hat{x}, \hat{\lambda}$ are C^k-functions.

Before discussing applications of this theorem we provide additional information about the bifurcating solutions. We have already seen $\hat{x}(s) = s\phi + o(s)$, but how does $\hat{\lambda}(s)$ behave exactly? Do we observe bifurcation from the left or from the right, or even none of those?

Corollary 4.6. *Let the assumptions of Theorem 4.4 hold and take $\psi \in Z' \setminus \{0\}$ as in Proposition 4.2. Then*

$$\hat{\lambda}'(0) = -\frac{1}{2}\frac{\psi(F_{xx}(\lambda_0, 0)[\phi, \phi])}{\psi(F_{x\lambda}(\lambda_0, 0)[\phi])}.$$

If $F \in C^3(\mathbb{R} \times X; Z)$ and $\hat{\lambda}'(0) = 0$ then

$$\hat{\lambda}''(0) = -\frac{1}{3}\frac{\psi(F_{xxx}(\lambda_0, 0)[\phi, \phi, \phi] - 3F_{xx}(\lambda_0, 0)[\phi, \zeta])}{\psi(F_{x\lambda}(\lambda_0, 0)[\phi])}$$

where $\zeta \in X$ is any solution of $F_x(\lambda_0, 0)[\zeta] = F_{xx}(\lambda_0, 0)[\phi, \phi]$.

Proof:
We exploit $\Phi(\hat{\lambda}(s), s) = 0$ for $|s| < \varepsilon$ where $\Phi(\lambda, s) = \int_0^1 \mathfrak{F}_s(\lambda, ts)\, dt$. So we get

$$\Phi_\lambda(\lambda_0, 0) = \mathfrak{F}_{s\lambda}(\lambda_0, 0) = \psi(F_{x\lambda}(\lambda_0, 0)[\phi]),$$
$$\Phi_s(\lambda_0, 0) = \frac{1}{2}\mathfrak{F}_{ss}(\lambda_0, 0) = \frac{1}{2}\psi(F_{xx}(\lambda_0, 0)[\phi, \phi]).$$

So the formula for $\hat{\lambda}'(0)$ follows from

$$0 = \frac{d}{ds}\Big|_{s=0}\Phi(\hat{\lambda}(s), s) = \Phi_\lambda(\lambda_0, 0)\hat{\lambda}'(0) + \Phi_s(\lambda_0, 0).$$

Now assume $\hat{\lambda}'(0) = 0$ and $F \in C^3$. Then

$$\begin{aligned} 0 &= \frac{d^2}{ds^2}\Big|_{s=0}\Phi(\hat{\lambda}(s), s) = \dots \\ &= \Phi_\lambda(\lambda_0, 0)\hat{\lambda}''(0) + \Phi_{\lambda\lambda}(\lambda_0, 0)\hat{\lambda}'(0)^2 + 2\Phi_{\lambda s}(\lambda_0, 0)\hat{\lambda}'(0) + \Phi_{ss}(\lambda_0, 0) \end{aligned}$$

$$= \Phi_\lambda(\lambda_0, 0)\hat{\lambda}''(0) + \Phi_{ss}(\lambda_0, 0).$$

So the formula for $\hat{\lambda}''(0)$ results from

$$\begin{aligned}
\Phi_{ss}(\lambda_0, 0) &= \frac{1}{3}\mathfrak{F}_{sss}(\lambda_0, 0) \\
&= \frac{1}{3}\psi(F_{xxx}(\lambda_0, 0)[\phi, \phi, \phi]) + \psi(F_{xx}(\lambda_0, 0)[\hat{y}_{ss}(\lambda_0, 0), \phi]) \\
&= \frac{1}{3}\psi(F_{xxx}(\lambda_0, 0)[\phi, \phi, \phi]) - \psi(F_{xx}(\lambda_0, 0)[\zeta, \phi]).
\end{aligned}$$

□

Note that in many cases the most complicated ζ-dependent term vanishes because of $F_{xx}(\lambda_0, 0) = 0$. In the context of nonlinear elliptic boundary value problems this occurs if the nonlinearity is "superquadratic" near zero.

End of Lec09

Corollary 4.7. *Assume H is a Hilbert space and $F \in C^2(\mathbb{R} \times H; H)$ satisfies $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Moreover assume*

(S') $F_x(\lambda_0, 0)$ is selfadjoint and is the sum of an invertible and a compact linear operator on H with $\ker(F_x(\lambda_0, 0)) = \text{span}\{\phi\}$,

(T') $\langle F_{x\lambda}(\lambda_0, 0)[\phi], \phi\rangle \neq 0$.

Then there is an $\varepsilon > 0$ and a curve $(\hat{\lambda}, \hat{x}) \in C^1((-\varepsilon, \varepsilon); \mathbb{R} \times H)$ such that $\hat{\lambda}(0) = \lambda_0, \hat{x}(0) = 0, \hat{x}'(0) = \phi$ and

$$F(\hat{\lambda}(s), \hat{x}(s)) = 0 \qquad \forall s \in (-\varepsilon, \varepsilon).$$

Moreover, in some open neighbourhood of $(\lambda_0, 0)$ all nontrivial solutions of $F(\lambda, x) = 0$ lie on this curve. Furthermore,

$$\hat{\lambda}'(0) = -\frac{1}{2}\frac{\langle F_{xx}(\lambda_0, 0)[\phi, \phi], \phi\rangle}{\langle F_{x\lambda}(\lambda_0, 0)[\phi], \phi\rangle}$$

and if $F \in C^3(X \times \mathbb{R}; Z)$ and $\hat{\lambda}'(0) = 0$, then

$$\hat{\lambda}''(0) = -\frac{1}{3}\frac{\langle F_{xxx}(\lambda_0, 0)[\phi, \phi, \phi] - 3F_{xx}(\lambda_0, 0)[\phi, \zeta], \phi\rangle}{\langle F_{x\lambda}(\lambda_0, 0)[\phi], \phi\rangle}$$

where $\zeta \in X$ is any solution of $F_x(\lambda_0, 0)[\zeta] = F_{xx}(\lambda_0, 0)[\phi, \phi]$.

Proof:
We show that Theorem 4.4 and Corollary 4.6 apply. Assumption (S') implies $F_x(\lambda_0, 0) = S + K$ where S is a linear invertible operator and K is linear and compact. Hence, $S^{-1}F_x(\lambda_0, 0) = I + S^{-1}K : H \to H$ and $S^{-1}K$ is linear and compact. Such operators are

known[21] to be Fredholm of index 0, so $F_x(\lambda_0,0)$ is Fredholm of index 0. As a consequence, $\ker(F_x(\lambda_0,0)) = \mathrm{span}\{\phi\}$ implies that $F_x(\lambda_0,0)$ is a $(1,1)$-Fredholm operator. In other words, assumption (S) holds.
Since $F_x(\lambda_0,0)$ is selfadjoint we have $\mathrm{ran}(F_x(\lambda_0,0)) = \ker(F_x(\lambda_0,0))^\perp$. So (T') implies

$$F_{x\lambda}(\lambda_0,0)[\phi] \notin \mathrm{span}\{\phi\}^\perp = \ker(F_x(\lambda_0,0))^\perp = \mathrm{ran}(F_x(\lambda_0,0)).$$

Hence, the transversality condition (T) holds and the Crandall-Rabinowitz Theorem applies. The bifurcation formulas for $\hat{\lambda}'(0), \hat{\lambda}''(0)$ follow from Corollary 4.6 because $\psi(z) = \langle z, \phi\rangle$ satisfies $\ker(\psi) = \mathrm{span}\{\phi\}^\perp = \mathrm{ran}(F_x(\lambda_0,0))$. □

Given that the assumptions simplify considerably, it is often advantageous to work in a Hilbert space setting (if one has the choice). We now provide applications of the Crandall-Rabinowitz Theorem.

Example 4.8. We consider the finite-dimensional problem given by

$$F : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n, \quad (\lambda,x) \mapsto A(\lambda)x - g(\lambda,x)$$

where $A(\cdot) \in C^2(\mathbb{R};\mathbb{R}^{n\times n}_{sym})$ and $g \in C^2(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$ satisfies $g(\lambda,0) = g_x(\lambda,0) = 0$ for all $\lambda \in \mathbb{R}$. Given that $\mathbb{R}^n$ is a Hilbert space and $F_x(\lambda_0,0) = A(\lambda_0)$ is selfadjoint, we need to check whether (S'),(T') satisfied.

- (S') $F_x(\lambda_0,0) = A(\lambda_0)$ is a index-zero Fredholm operator on $\mathbb{R}^n$ because any bounded linear operator on $\mathbb{R}^n$ is Fredholm (why?). So the simplicity condition (S') is satisfied if and only if $\ker(A(\lambda_0)) = \mathrm{span}\{\phi\}$ for some $\phi \in \mathbb{R}^n \setminus \{0\}$.
- (T') We have $F_x(\lambda,x)[\phi] = A(\lambda)\phi - g_x(\lambda,x)[\phi]$ and, due to $g_{x\lambda}(\lambda,0) = 0$ for all $\lambda\in\mathbb{R}$, $F_{x\lambda}(\lambda_0,0)[\phi] = A'(\lambda_0)\phi$. (The entries of the matrix $A'(\lambda_0)$ are the derivatives of the corresponding entries of $A(\lambda)$ at $\lambda = \lambda_0$.) So the transversality condition holds if and only if $\langle\phi, A'(\lambda_0)\phi\rangle \neq 0$.

Example 4.9. Let $\Gamma \in L^\infty(\Omega)$ and $N \in \{1,2,3\}$[22] Let $\lambda_0 > 0$ be a simple eigenvalue of the Laplacian in $H^1_0(\Omega)$. (The existence of (not necessarily simple) eigenvalues is ensured by the spectral theory of compact selfadjoint operators in Hilbert spaces.) We show that $(\lambda_0,0)$ is a bifurcation point for

$$-\Delta u - \lambda u = \Gamma(x)u^3 \qquad (u \in H^1_0(\Omega)).$$

To prove this we reformulate the equation as

$$F(\lambda,u) = 0 \qquad \text{where } F(\lambda,u) := u - \lambda(-\Delta)^{-1}u - (-\Delta)^{-1}(\Gamma u^3).$$

[21] see for instance [2, p.169 (c)]

[22] This assumption implies that $u \mapsto (-\Delta)^{-1}(\Gamma u^3)$ is a well-defined map from $H^1_0(\Omega)$ to itself, and in fact even twice continuously differentiable. For regular Γ and Ω one may alternatively work in $C^{0,\alpha}(\overline{\Omega})$ using "Schauder estimates" for the Laplacian, see Chapter 6 in [7].

Then $F \in C^2(\mathbb{R} \times H_0^1(\Omega); H_0^1(\Omega))$ and the linear operator $F_u(\lambda_0, 0)[h] = h - \lambda_0(-\Delta)^{-1}h$ is selfadjoint[23] and Fredholm because $(-\Delta)^{-1}$ is compact from $H_0^1(\Omega)$ to itself[24]. Moreover, $\ker(F_u(\lambda_0, 0)) = \mathrm{span}\{\phi\}$ and

$$\langle F_{u\lambda}(\lambda_0, 0)[\phi], \phi\rangle = \langle -(-\Delta)^{-1}\phi, \phi\rangle = -\frac{1}{\lambda_0}\|\phi\|^2 < 0.$$

So the Crandall-Rabinowitz Theorem applies.

Let us compute the direction of bifurcation. We have for $\phi_1, \phi_2, \phi_3 \in H_0^1(\Omega)$

$$\begin{aligned} F_{uu}(\lambda, u)[\phi_1, \phi_2] &= -6(-\Delta)^{-1}(\Gamma u \phi_1 \phi_2), \\ F_{uuu}(\lambda, u)[\phi_1, \phi_2, \phi_3] &= -6(-\Delta)^{-1}(\Gamma \phi_1 \phi_2 \phi_3). \end{aligned}$$

From Corollary 4.6 we thus get $\hat{\lambda}'(0) = 0$ as well as

$$\begin{aligned} \hat{\lambda}''(0) &= -\frac{1}{3}\frac{\langle -6(-\Delta)^{-1}(\Gamma\phi^3), \phi\rangle}{-\frac{1}{\lambda_0}\|\phi\|^2} \\ &= -2\lambda_0\|\phi\|^{-2}\langle(-\Delta)^{-1}(\Gamma\phi^3), \phi\rangle \\ &= -2\lambda_0\|\phi\|^{-2}\int_\Omega (\Gamma(x)\phi(x)^3)\cdot\phi(x)\,dx \\ &= -2\lambda_0\|\phi\|^{-2}\int_\Omega \Gamma(x)\phi(x)^4\,dx. \end{aligned}$$

Consequence: For positive Γ the branches turn to the left, for negative Γ to the right. What happens in the linear case $\Gamma \equiv 0$? We have $\hat{\lambda}(s) = \lambda, \hat{x}(s) = s\phi$, which is consistent with $\hat{\lambda}'(0) = \hat{\lambda}''(0) = 0$.

End of Lec10

We comment on the simplicity assumption in the context of boundary value problems for ODEs respectively PDEs. In the case of ordinary differential equations the simplicity of all(!) eigenvalues is usually easy to establish. Consider for example some Sturm-Liouville-type boundary value problem of Dirichlet type

$$-(py')' - \lambda c y = 0, \qquad y(0) = y(T) = 0. \tag{4.3}$$

[23]It suffices to show that $(-\Delta)^{-1} : H_0^1(\Omega) \to H_0^1(\Omega)$ is symmetric. Recall $v := (-\Delta)^{-1}u$ is, by definition, equivalent to

$$\langle v, \phi\rangle_{H_0^1(\Omega)} = \int_\Omega u\phi\,dx \qquad \text{for all } \phi \in H_0^1(\Omega).$$

Hence, we have for all $\phi \in H_0^1(\Omega)$

$$\langle(-\Delta)^{-1}u, \phi\rangle_{H_0^1(\Omega)} = \langle u, \phi\rangle_{L^2(\Omega)} = \langle \phi, u\rangle_{L^2(\Omega)} = \langle(-\Delta)^{-1}\phi, u\rangle_{H_0^1(\Omega)}.$$

[24]Indeed, if $(f_n) \subset H_0^1(\Omega)$ is bounded, then a subsequence (f_{n_k}) is convergent in $L^2(\Omega)$ with limit $f \in L^2(\Omega)$ because of the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ (Rellich-Kondrachov Theorem). But then the boundedness = continuity of $(-\Delta)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ implies $(-\Delta)^{-1}f_{n_k} \to (-\Delta)^{-1}f$, which is all we had to prove.

Can this linear problem have a two-/higher-dimensional solution space? Under usual assumptions the answer is no: Take a positive function $p \in C^1([0,T])$ and $c \in C([0,T])$. The Picard-Lindelöf Theorem implies that each solution of (4.3) can be written as $y(x) = \alpha y_1(x) + \beta y_2(x)$ for all $x \in [0,T]$ where y_1, y_2 solve the ODE with initial conditions $y_1(0) = 0, y_1'(0) = 1$ and $y_2(0) = 1, y_2'(0) = 1$. The boundary conditions $y(0) = y_1(0) = 0 \neq y_2(0)$ imply $\beta = 0$ and thus $y \in \operatorname{span}\{y_1\}$. So the solution space of (4.3) is one-dimensional.

In the context of elliptic PDEs such a strong result does not hold. But one has the Krein-Rutman Theorem [10] (see Section 3.6.3 in [3]) that allows to say that the first eigenvalue is simple.

Theorem 4.10 (see Corollary 3.6.13 in [3]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $c \in C(\overline{\Omega})$ positive. Then the eigenvalue problem*

$$-\Delta u = \lambda c(x) u \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega) \tag{4.4}$$

has a positive eigenfunction with positive simple eigenvalue $\lambda_1 = \min \sigma(-c(x)^{-1}\Delta) > 0$.

To circumvent the simplicity assumption it is sometimes possible to perform the analysis in a smaller space in which the given eigenvalue is simple. For instance, if the eigenvalue problem (4.4) is considered on a ball $\Omega = \{x \in \mathbb{R}^N : |x| < R\}$ then most of the eigenvalues in $H_0^1(\Omega)$ will be non-simple. If c is radially symmetric as well, i.e. $c(x) = c_0(|x|)$, then it makes sense to reconsider the eigenvalue problem in the smaller space

$$\begin{aligned} H^1_{0,\mathrm{rad}}(\Omega) &= \{u \in H_0^1(\Omega) : u(Qx) = u(x) \quad \forall Q \in O(N)\ \forall x \in \Omega\} \\ &= \overline{\{u_0(|\cdot|) : u_0'(0) = 0\}}^{\|\cdot\|} \end{aligned}$$

where the closure is taken with respect to the norm

$$\|\phi\| := \left(\int_0^R r^{N-1} (\phi'(r)^2 + \phi(r)^2)\, dr \right)^{1/2}.$$

A similar approach is possible for annuli. Functions belonging to $H^1_{0,\mathrm{rad}}(\Omega)$ are given by some profile function $u_0 : [0,R] \to \mathbb{R}$, which reduces the PDE eigenvalue problem to some singular ODE eigenvalue problem. In fact, (4.4) then corresponds to

$$-u_0''(r) - \frac{N-1}{r} u_0'(r) = \lambda c_0(r) u_0(r), \qquad u_0'(0) = 0,\ u_0(R) = 1.$$

5 Bifurcation from Infinity

Definition 5.1. *We say that bifurcation from ∞ occurs at $\lambda = \lambda_0$ for the equation $F(\lambda, x) = 0$ if there is a sequence (μ_n, x_n) in $\mathbb{R} \times X$ such that $F(\mu_n, x_n) = 0$ and $\|x_n\|_X \to \infty, \mu_n \to \lambda_0$ as $n \to \infty$.*

The following result basically goes back to Rabinowitz [11], but the proof is essentially taken from [9, Theorem I.20.1].

Theorem 5.2. *Let X, Z be a real Banach space and let $F : \mathbb{R} \times X \to Z$ be given by*

$$F(\lambda, x) = L(\lambda)x + R(\lambda, x)$$

for $L \in C^1(\mathbb{R}, \mathcal{L}(X; Z))$ and $R \in C^1(\mathbb{R} \times X; Z)$ having the following properties:

(S) $L(\lambda_0)$ *is* $(1,1)$*-Fredholm,*

(T) $L'(\lambda_0)[\phi] \notin \operatorname{ran}(L(\lambda_0))$ *where* $\ker(L(\lambda_0)) = \operatorname{span}\{\phi\}$,

(R) There is an open neighbourhood $U \subset \mathbb{R} \times X \times \mathbb{R}$ of $(\lambda_0, \phi, 0)$ such that the function $(\lambda, x, s) \mapsto sR(\lambda, s^{-1}x)$ admits an extension $\mathcal{R} \in C^1(U; Z)$ such that

$$\mathcal{R}(\lambda_0, \phi, 0) = 0, \quad \mathcal{R}_\lambda(\lambda_0, \phi, 0) = 0, \quad \mathcal{R}_x(\lambda_0, \phi, 0) = 0.$$

Then bifurcation from infinity occurs at $\lambda = \lambda_0$. More precisely, there is a continuously differentiable curve $(\hat{\lambda}, \hat{x})$ satisfying $\hat{\lambda}(0) = \lambda_0, s\hat{x}(s) \to \phi$ as $s \to 0$ and $F(\hat{\lambda}(s), \hat{x}(s)) = 0$.

Proof:
We choose ψ as in Proposition 4.2 such that $\ker(\psi) = \operatorname{ran}(L(\lambda_0))$. Our aim is to apply the Implicit Function Theorem to the augmented equation $\mathfrak{F}(\lambda, x) = 0$ where

$$\mathfrak{F} : U \to Z \times \mathbb{R}, \quad (\lambda, x, s) \mapsto \left(L(\lambda)x + \mathcal{R}(\lambda, x, s), \psi(L'(\lambda_0)[x - \phi])\right).$$

Note that $L(\lambda)x + \mathcal{R}(\lambda, x, s) = 0$ is equivalent to $sF(\lambda, s^{-1}x) = 0$, so any zero (λ, x, s) of $\mathfrak{F}$ produces a nontrivial solution $(\lambda, s^{-1}x)$ of our problem. By assumption (R) we have $\mathfrak{F} \in C^1(U; Z)$. Moreover, $\mathfrak{F}(\lambda_0, \phi, 0) = (0, 0)$ and

$$\mathfrak{F}_\lambda(\lambda_0, \phi, 0) = \left(L'(\lambda_0)\phi, 0\right), \qquad \mathfrak{F}_x(\lambda_0, \phi, 0)[h] = \left(L(\lambda : 0)h, \psi(L'(\lambda_0)[h])\right).$$

In order to solve for (λ, x) in terms of s we need to check that $\partial_{(\lambda,x)}\mathfrak{F}(\lambda_0, \phi, 0) : \mathbb{R} \times X \to Z \times \mathbb{R}$ is invertible.

Let $(z, \sigma) \in Z \times \mathbb{R}$. We are looking for $(\mu, h) \in \mathbb{R} \times X$ such that $\mathfrak{F}_x(\lambda_0, \phi, 0)[h] + \mu\mathfrak{F}_\lambda(\lambda_0, \phi, 0)[h] = (z, \sigma)$, so

$$(L(\lambda_0)[h], \psi(L'(\lambda_0)[h])) + \mu(L'(\lambda_0)[\phi], 0) = (z, \sigma). \tag{5.1}$$

This is equivalent to

$$L(\lambda_0)[h] = z - \mu L'(\lambda_0)[\phi], \qquad \psi(L'(\lambda_0)[h]) = \sigma$$

We decompose $h \in X = \ker(L) \oplus \tilde{X}$ as in (4.1) into $h = s\phi + \tilde{x}$ for $s \in \mathbb{R}, \tilde{x} \in \tilde{X}$. This gives the equivalent system

$$L(\lambda_0)[\tilde{x}] = z - \mu L'(\lambda_0)[\phi], \qquad s\psi(L'(\lambda_0)[\phi]) + \psi(L'(\lambda_0)[\tilde{x}]) = \sigma. \tag{5.2}$$

Applying ψ to the first equation in (5.2) we find

$$\mu = \frac{\psi(z)}{\psi(L'(\lambda_0)[\phi])}, \qquad \tilde{x} = L(\lambda_0)|_{\tilde{X}}^{-1}(z - \mu L'(\lambda_0)[\phi]).$$

Note that $L(\lambda_0)$ is invertible as a map from $\tilde{X}$ to $\operatorname{ran}(L(\lambda_0))$, which makes $\tilde{x} \in \tilde{X}$ well-defined. In view of $\psi(L'(\lambda_0)[\phi]) \neq 0$ the second equation in (5.2) is uniquely solvable via

$$s = \frac{\sigma - \psi(L'(\lambda_0)[\tilde{x}])}{\psi(L'(\lambda_0)[\phi])}.$$

Conclusion: (5.1) has a uniquely determined solution $(\mu, h) \in \mathbb{R} \times \tilde{X}$. This proves the invertibility of $\partial_{(\lambda,x)}\mathfrak{F}(\lambda_0, \phi, 0)$.

So the Implicit Function Theorem 3.5 with $k = 1$ yields a continuously differentiable function $(\hat{\lambda}, \hat{y}) : (-\varepsilon, \varepsilon) \to \mathbb{R} \times X$ satisfying $\hat{\lambda}(0) = \lambda_0, \hat{y}(0) = \phi$ as well as $\mathfrak{F}(\hat{\lambda}(s), \hat{y}(s), s) = 0$ for $|s| > 0$ sufficiently small. Now set $\hat{x}(s) := s^{-1}\hat{y}(s)$ for $s \neq 0$. Then $F(\hat{\lambda}(s), \hat{x}(s)) = 0$ holds for all $s \neq 0$ and $s\hat{x}(s) \to \phi, \hat{\lambda}(s) \to \lambda_0$ as $s \to 0$. □

Example 5.3. We consider the problem

$$A(\lambda)x = \frac{x}{1 + |x|^2} \qquad (x \in \mathbb{R}^n)$$

where $A \in C^1(\mathbb{R}; \mathbb{R}^{n\times n}_{sym})$. Assume $\ker(A(\lambda_0)) = \operatorname{span}\{\phi\}$ so that (S) holds. The transversality condition holds provided that $\langle A'(\lambda_0)\phi, \phi\rangle \neq 0$. The proof is the same as in Corollary 4.7. Furthermore, we have

$$\mathcal{R}(\lambda, x, s) := s \cdot \frac{\frac{x}{s}}{1 + |\frac{x}{s}|^2} = \frac{s^2 x}{s^2 + |x|^2}.$$

This function is smooth[25] in a neighbourhood of $(\lambda_0, \phi, 0)$ and (R) holds. So the theorem shows:

$$\ker(A(\lambda_0)) = \operatorname{span}\{\phi\} \text{ and } \langle A'(\lambda_0)\phi, \phi\rangle \neq 0 \quad \Rightarrow \quad \text{Bifurcation from infinity at } \lambda_0.$$

Example 5.4. Consider the nonlinear boundary value problem

$$-u'' - \lambda u = \frac{a(x)u}{1 + b(x)u^4} \qquad u \in C^2(\mathbb{R}) \text{ is } 2\pi - \text{periodic}$$

for[26] 2π-periodic functions $a, b \in C(\mathbb{R})$ such that b is positive. As Banach spaces we choose $X = C^2_{per}, Z = C_{per}$ where

$$C^k_{per} = \{u \in C^k(\mathbb{R}) : u \text{ is } 2\pi - \text{periodic}\} \qquad (k \in \mathbb{N}_0).$$

Define $L(\lambda)u := u''\lambda u$ and $R(u, \lambda) := \frac{au}{1+bu^4}$. We shall prove bifurcation from infinity at $\lambda_0 = 0$.

Claim: $L(\lambda_0)$ is a (1,1)-Fredholm operator with $\ker(L(\lambda_0)) = \text{span}\{1\}$.

Proof: $L(\lambda_0)u = 0$, $u \in X$ is equivalent to u being a 2π-periodic affine function, so $\ker(L(\lambda_0)) = \text{span}\{\phi\}$ where $\phi(x) = 1$ for all $x \in \mathbb{R}$. On the other hand, $f := u''$ with $u \in X$ satisfies $\int_0^{2\pi} f(t)\,dt = u'(2\pi) - u'(0) = 0$, which shows $\text{ran}(L(\lambda_0)) \subset \{f \in Z : \int_0^{2\pi} f(t)\,dt = 0\}$. The reverse inclusion results from the choice

$$u(x) := \int_0^x (x - t) f(t)\,dt - \frac{x}{2\pi} \int_0^{2\pi} t f(t)\,dt$$

because of $Lu = -u'' = f$ and $u(2\pi) = u(0), u'(2\pi) = u'(0)$. Indeed, the latter implies $u(x+2\pi) = u(x)$ for all $x \in \mathbb{R}$ via the Picard-Lindelöf Theorem for the initial value problem $y'' = f, y(0) = u(0), y'(0) = u'(0)$, so $u \in X$. We have thus shown that $\ker(L(\lambda_0))$ is one-dimensional and $\text{ran}(L(\lambda_0))$ has codimension one.

So assumption (S) holds. The transversality condition requires $L'(\lambda_0)[\phi] \notin \text{ran}(L(\lambda_0))$, i.e. $\phi \notin \text{ran}(L(\lambda_0))$. This is true because $\int_0^{2\pi} \phi(x)\,dx = 2\pi \neq 0$. So assumption (T) holds as well. As to assumption (R)

$$\mathcal{R}(\lambda, u, s) := s \cdot \frac{\frac{au}{s}}{1 + b|\frac{u}{s}|^4} = \frac{s^4 au}{s^4 + bu^4}. \tag{5.3}$$

This function is smooth in a neighbourhood of $(\lambda_0, \phi, 0)$. So the theorem applies and we observe bifurcation from infinity.

We remark that Theorem 5.2 does not provide a uniqueness statement, which comes from the fact that the "phase condition" $\psi(L'(\lambda_0)[x - \phi]) = 0$ is somewhat arbitrary and may be modified. Our applications do not treat boundary value problems where the corresponding solutions have zeros. In fact, the regularity assumption (R) does typically not hold in this case as in Example 5.4: The function $\mathcal{R}$ from (5.3) is not continuously

[25] It is even smooth on $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$, but not close to $(\lambda_0, 0, 0)$.

[26] Note that we have to change the functional setup if a or b is discontinuous.

differentiable at $(s,u)=(0,0)$. Nevertheless, bifurcation from infinity can also be proved in such cases, but with different methods based on degree theory (Rabinowitz' Global Bifurcation Theorem) that does not require $\mathcal{R}\in C^1$, but weaker hypotheses. This approach is the original found by Rabinowitz [11].

End of Lec12

6 Digression: The Ljapunov-Schmidt reduction

The Ljapunov-Schmidt reduction is a fundamental tool in bifurcation theory. We have already used it in the context of the Crandall-Rabinowitz Theorem. The ultimate goal is to find nontrivial solutions for equations of the form $F(\lambda,x)=0$ where $x\in X,\lambda\in\mathbb{R}$ and $F\in C^1(\mathbb{R}\times X;Z)$- If $L:=F_x(\lambda_0,0):X\to Z$ is a (p,p)-Fredholm operator then we have

$$X=\ker(L)\oplus\tilde{X},\qquad Z=\operatorname{ran}(L)\oplus\tilde{Z}$$

where[27] $\dim(\ker(L))=\dim(\tilde{Z})=p\in\mathbb{N}_0$.

What is the advantage of such a decomposition? We want to exploit that the operator L is "degenerate" only on a finite-dimensional subspace, namely on $\ker(L)$. More precisely, if L is considered as a map from $\tilde{X}$ to $\operatorname{ran}(L)$ then it is a bounded linear and invertible operator. So the operator $L|_{\tilde{X}}:\tilde{X}\to\operatorname{ran}(L)$ has a well-defined bounded linear inverse. In the context of the Crandall-Rabinowitz Theorem we used this fact in Lemma 4.3 to solve $F(x,\lambda)=0$ up to some one-dimensional (!) equation

$$\psi(F(\lambda,\hat{y}(\lambda,s)+s\phi))=0\qquad s,\lambda\in\mathbb{R}.$$

We recall that $s\phi$ parameterizes $\ker(L)$ and $\hat{y}(\lambda,s)\in\tilde{X}$ was found with the Implicit Function Theorem. The same can be done in the general case. To see this choose a bounded linear projector $P:Z\to\operatorname{ran}(L)$ such that $\ker(P)=\tilde{Z}$. (The projector is constructed with the aid of the Hahn-Banach Theorem.) Then the equation $PF(x,\lambda)=0$ can be recast as the system

$$PF(\lambda,y+\psi)=0,\quad (I-P)F(\lambda,y+\psi)=0\qquad\text{where }\psi\in\ker(L),y\in\tilde{X},\lambda\in\mathbb{R}.\tag{6.1}$$

We (have to) content ourselves to solving the first equation with the aid of the Implicit Function Theorem, so define

$$G:\mathbb{R}\times\ker(L)\times\tilde{X}\to\operatorname{ran}(L),\quad(\lambda,\psi,y)\mapsto PF(\lambda,y+\psi).$$

This function is continuously differentiable and its partial derivative with respect to y, i.e., $G_y(\lambda_0,0,0):\tilde{X}\to\operatorname{ran}(L)$, $h\mapsto PF_x(\lambda_0,0)[h]=Lh$ is invertible. As a consequence,

[27] In the special case of Hilbert spaces X,Z one may choose $\tilde{X}=\ker(L)^\perp$ and $\tilde{Z}=\operatorname{ran}(L)^\perp$, and if $X=Z$ and $L:X\to X$ is selfadjoint then the whole decomposition can be rewritten as $X=\ker(L)\oplus_\perp\operatorname{ran}(L)$, i.e., $\tilde{X}=\operatorname{ran}(L),\tilde{Z}=\ker(L)$ are the usual choices.

we can solve $G(\lambda, \psi, y) = 0$ in a neighbourhood of the point $(\lambda_0, 0, 0)$ for the variable y and obtain $y = \hat{y}(\lambda, \psi)$. To solve (6.1), it thus remains to solve

$$(I - P)F(\lambda, \hat{y}(\lambda, \psi) + \psi) = 0 \quad \text{where } \psi \in \ker(L), \lambda \in \mathbb{R}. \tag{6.2}$$

We stress that this is a finite-dimensional problem: if $\dim(\ker(L)) = p$ then $\dim((I - P)Z) = \dim(\tilde{Z}) = p$, so this is a system of p (typically nonlinear) equations for $p + 1$ unknowns $(\lambda, \psi) \in \mathbb{R} \times \ker(L)$ in the sense of one $(p+1)$-dimensional unknown. Usually, (6.2) is easier to analyze even though $\hat{y}(\lambda, \psi)$ is almost never explicitly known. On the other hand, one knows that ψ is the "leading order term" in this equation because of $\hat{y}_\psi(\lambda_0, 0) = 0$. For more information see for instance [3, p.33-34].

Conclusion: For $(\lambda, x) \approx (\lambda_0, 0)$ solving $F(x, \lambda) = 0$ is equivalent to solving the finite-dimensional equation (6.2). This procedure is called the Ljapunov-Schmidt reduction.

Remark 6.1.

(a) The equation (6.2) is p-dimensional (because $\dim(\tilde{Z}) = p$) and it has a $(p + 1)$-dimensional unknown, namely the p-dimensional unknown $\psi \in \ker(L)$ and the one-dimensional unknown $\lambda \in \mathbb{R}$. So the system gets larger, the bigger the p is. The Crandall-Rabinowitz Theorem dealing with $p = 1$ may therefore be regarded as the easiest case.

(b) Up to the construction of the projectors on $\tilde{X}, \tilde{Z}$, there is nothing special about finite dimensions: one may equally reduce $F(\lambda, x) = 0$ to a reduced infinite-dimensional equation (6.2). The question is whether this makes the theory easier.

7 Hopf Bifurcation

In this section we prove a bifurcation result for periodic solutions of ODEs

$$y'(t) = f(\lambda, y(t)) \qquad y \in C^1(\mathbb{R}; X), \lambda \in \mathbb{R} \tag{7.1}$$

where $f : \mathbb{R} \times X \to X$ is sufficiently smooth and X is a real Banach space. In order to keep the technicalities at a reasonable level we concentrate on the case $X = \mathbb{R}^n$ with $n \in \mathbb{N}$. The more general discussion can be found in [5]. In contrast to earlier examples like Example 5.4 the period is not fixed a priori, but the solutions bifurcating from the trivial solution family will have different periods. This phenomenon is called Hopf bifurcation in honor of its discoverer Eberhard Hopf [8].

We will see that the following assumptions guarantee the existence of small periodic solutions bifurcating from the trivial solution family:

(H1) ("Regularity") $f \in C^2(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$ with $f(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

(H2) ("Simplicity") There are $\beta > 0$ and $\phi, \psi \in \mathbb{C}^n$ such that $\langle \psi, \phi \rangle_{\mathbb{C}^n} = 1$ and[28]

$$\ker(f_x(\lambda_0, 0) - i\beta) = \mathrm{span}_{\mathbb{C}}\{\phi\}, \qquad \ker(f_x(\lambda_0, 0)^T + i\beta) = \mathrm{span}_{\mathbb{C}}\{\psi\}$$

(H3) ("Nonresonance") For all $k \in \mathbb{Z}, |k| \neq 1$ we have $\ker(f_x(\lambda_0, 0) - ik\beta) = \{0\}$.

(H4) ("Transversality") $\Re\left(\langle f_{x\lambda}(\lambda_0, 0)[\phi], \psi \rangle_{\mathbb{C}^n}\right) \neq 0$.

Here: $\langle \phi, \psi \rangle_{\mathbb{C}^n} := \sum_{i=1}^n \phi_i \overline{\psi_i}$.

Before stating the main theorem we provide an auxiliary result that will be needed in the proof. It concerns the computation of integrals

$$Y_k(t) := -\frac{1}{\beta} \int_0^t \underbrace{\exp\left(\frac{t-\tau}{\beta} A\right)}_{\in \mathbb{R}^{n \times n}} \xi \, e^{ik\tau} \, d\tau \in \mathbb{C}^n$$

where $k \in \mathbb{Z}$ and $\xi \in \mathbb{C}^n$, $A := f_x(\lambda_0, 0) \in \mathbb{R}^{n \times n}$.

Proposition 7.1. *Let $\xi \in \mathbb{C}^n$. Then, for ϕ, ψ as in (H2), we get*

(i) If $|k| \neq 1$ then

$$Y_k(t) = \left(e^{ikt} I - \exp\left(\frac{t}{\beta} A\right)\right)(A - ik\beta)^{-1} \xi.$$

(ii) If $k \in \{-1, 1\}$ then $Y_k(t)$ is given by

$$Y_1(t) = \left(e^{it} I - \exp\left(\frac{t}{\beta} A\right)\right) w_1 - \frac{t}{\beta} e^{it} \langle \xi, \psi \rangle_{\mathbb{C}^n} \phi,$$

[28] This automatically implies $\psi \cdot \phi = 0$.

$$Y_{-1}(t) = \left(e^{-it}I - \exp\left(\frac{t}{\beta}A\right)\right)w_{-1} - \frac{t}{\beta}e^{-it}\langle\xi,\bar{\psi}\rangle_{\mathbb{C}^n}\bar{\phi},$$

where $w_1, w_{-1} \in \mathbb{C}^n$ are uniquely determined via

$$(A - i\beta)w_1 = \xi - \langle\xi,\psi\rangle_{\mathbb{C}^n}\phi \in \operatorname{ran}(A - i\beta),$$
$$(A + i\beta)w_{-1} = \xi - \langle\xi,\bar{\psi}\rangle_{\mathbb{C}^n}\bar{\phi} \in \operatorname{ran}(A + i\beta).$$

End of Lec13

In order to work with a fixed period, say 2π, we introduce an auxiliary frequency parameter $\omega = \frac{T}{2\pi}$ and rescale the equation accordingly: y^* is a T-periodic solution of (7.1) if and only if $y(t) := y^*(\omega t)$ is a 2π-periodic solutions of the ODE

$$y'(t) = \omega f(\lambda, y(t)). \tag{7.2}$$

In order to single out the linear term at $\lambda = \lambda_0$ we write $f(\lambda, y(t)) = Ay(t) + R(t)$. So we have to solve

$$y'(t) = \omega Ay(t) + \omega R(t).$$

The variation of constants formula implies that it is sufficient to look for continuous and 2π-periodic solutions y of the integral equation $F(y,\omega,\lambda) = 0$ where

$$F(y,\omega,\lambda)(t) := y(t) - e^{\omega tA}y(0) - \omega\int_0^t e^{\omega(t-\tau)A}\underbrace{(f(\lambda,y(\tau)) - Ay(\tau))}_{=R(\tau)}\,d\tau.$$

Here, $A = f_x(\lambda_0, 0) \in \mathbb{R}^{n\times n}$ as above. We shall prove the following.

Theorem 7.2 (Hopf, 1943)**.** *Let $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfy (H1)-(H4). Then there are C^1-curves $\hat{\lambda}, \hat{T} : (-\varepsilon,\varepsilon) \to \mathbb{R}$ and $\hat{y} : (-\varepsilon,\varepsilon) \to C_{per}(\mathbb{R};\mathbb{R}^n)$ such that $\hat{y}(0) = 0, \hat{\lambda}(0) = \lambda_0, \hat{T}(0) = \frac{2\pi}{\beta}$ and, for $0 < |s| < \varepsilon$, the function $t \mapsto \hat{y}(s)(2\pi t/\hat{T}(s))$ is a nonconstant $\hat{T}(s)$-periodic C^1−solution of (7.1) for $\lambda = \hat{\lambda}(s)$.*
Moreover, every T-periodic solution (y,λ,T) of this equation lying in a small neighbourhood of $(0,\lambda_0,\frac{2\pi}{\beta})$ is of this form up to translations.

Proof. Define

$$Y := \{y \in C([0,2\pi];\mathbb{R}^n) : y(0) = y(2\pi)\},$$
$$Z := \{z \in C([0,2\pi];\mathbb{R}^n) : z(0) = 0\}.$$

With these definitions we have $F \in C^2(Y\times\mathbb{R}\times\mathbb{R}; Z)$ by (H1). For $\omega_0 := \frac{1}{\beta}$ we have $F(0,\omega_0,\lambda_0) = 0$. The linearized operator with respect to y at this solution is given by

$$F_y(0,\omega_0,\lambda_0)[h](t) = h(t) - e^{\omega_0 At}h(0) \qquad (h \in Y, t \in \mathbb{R}).$$

We shall later prove (Step II)

$$\ker(F_y(0,\omega_0,\lambda_0)) = \mathrm{span}_{\mathbb{R}}\{y_1,y_2\} \qquad \text{where} \quad y_1(t) = \Re(e^{it}\phi),\ y_2(t) = \Im(e^{it}\phi). \tag{7.3}$$

Similar to the proof of the Crandall-Rabinowitz Theorem we define[29] V to be a complement of $\mathrm{span}\{y_1,y_2\}$ in Y and $\hat{F} \in C^1(\mathbb{R}\times V\times\mathbb{R}\times\mathbb{R};Z)$ via

$$\begin{aligned}\hat{F}(s,v,\omega,\lambda) &:= \int_0^1 F_y(s\tau(y_1+v),\omega,\lambda)[y_1+v]\,d\tau \\ &= \begin{cases}\frac{1}{s}F(s(y_1+v),\omega,\lambda) & ,\text{if } s\neq 0\\ F_y(0,\omega,\lambda)[y_1+v] & ,\text{if } s=0.\end{cases}\end{aligned}$$

We use without proof that one reasonable choice[30] for V is

$$V = \left\{v\in Y : \int_0^{2\pi} v(\tau)\cdot y_1(\tau)\,d\tau = \int_0^{2\pi} v(\tau)\cdot y_2(\tau)\,d\tau = 0\right\},$$

and this space is translation-invariant if v is identified with its 2π-periodic extension[31]. So (7.3) implies $\hat{F}(0,0,\omega_0,\lambda_0) = 0$. Below we will show that the partial derivative $\hat{F}_{(v,\omega,\lambda)}(0,0,\omega_0,\lambda_0)$ is bijective so that the Implicit Function theorem provides a C^1-curve $(\hat{v},\hat{\omega},\hat{\lambda}) : (-\varepsilon,\varepsilon)\to V\times\mathbb{R}\times\mathbb{R}$ such that $\hat{v}(0)=0, \hat{\omega}(0)=\omega_0, \hat{\lambda}(0)=\lambda_0$ and

$$\hat{F}(s,\hat{v}(s),\hat{\omega}(s),\hat{\lambda}(s)) = 0 \quad \text{for } |s|<\varepsilon.$$

Then $\hat{y}(s) := s(y_1+\hat{v}(s))$ is a nontrivial 2π-periodic solution of $y' = \hat{\omega}(s)f(\hat{\lambda}(s),y)$ being a zero of F satisfying $\hat{y}(s)(0) = \hat{y}(s)(2\pi)$. In fact, the latter implies $\hat{y}(s)(t) = \hat{y}(s)(2\pi+t)$ for all $t\in\mathbb{R}$ via the Picard-Lindelöf-Theorem. Hence we get $\hat{T}(s)$-periodic solutions $\hat{y}^*(s)$ of (7.1) by putting

$$\hat{y}^*(s)(t) := \hat{y}(s)(t\hat{\omega}(s)^{-1}), \qquad \hat{T}(s) := 2\pi\hat{\omega}(s).$$

In view of $s^{-1}\hat{y}(s)\to y_1\neq 0$ as $s\to 0$ these solutions are nontrivial for $0<|s|<\varepsilon$ and $\hat{T}(0) = 2\pi\hat{\omega}(0) = \frac{2\pi}{\beta}$. It remains to check:

- the uniqueness up to translations (Step VII),
- the formula (7.3) (Step II),
- the bijectivity of $F_{(v,\omega,\lambda)}(0,0,\omega_0,\lambda_0)$ (Step VI). **End of Lec14**

[29] It is remarkable that $\hat{F}$ is not defined on the full space, but only on $\mathrm{span}\{y_1\}\oplus V$, which has codimension one. (The space $\mathrm{span}\{y_2\}$ is missing.) One can check that the analogous analysis does not work on Y. For instance, the linearized operator is not injective any more.

[30] In the general case of Hopf bifurcation in infinite-dimensional Banach spaces, this definition needs to be modified.

[31] In other words, if v is continuous and 2π–periodic and $v\in V$ implies $v(\cdot+\theta)\in V$ for all $\theta\in\mathbb{R}$. Indeed, exploiting that $y_1(\tau-\theta) = \cos(\theta)y_1(\tau)+\sin(\theta)y_2(\tau)$ is 2π-periodic, we get for all $\theta\in\mathbb{R}$

$$\begin{aligned}\int_0^{2\pi} v(\tau+\theta)\cdot y_1(\tau)\,d\tau &= \int_\theta^{2\pi+\theta} v(\tau)\cdot y_1(\tau-\theta)\,d\tau = \int_0^{2\pi} v(\tau)\cdot y_1(\tau-\theta)\,d\tau \\ &= \cos(\theta)\int_0^{2\pi} v(\tau)\cdot y_1(\tau)\,d\tau + \sin(\theta)\int_0^{2\pi} v(\tau)\cdot y_2(\tau)\,d\tau \\ &= 0\end{aligned}$$

Step I: Kernel and range for $L := \mathrm{id} - \exp(\frac{2\pi}{\beta}A)$. We show[32]

$$\ker(L) = \mathrm{span}\{\mathfrak{R}(\phi), \mathfrak{I}(\phi)\}, \qquad \mathrm{ran}(L) = \{\mathfrak{R}(\psi), \mathfrak{I}(\psi)\}^{\perp}.$$

We prove the first equality. The inclusion "$\supset$" follows from[33]

$$\begin{aligned}\exp(\frac{2\pi}{\beta}A)\mathfrak{R}(\phi) &= \mathfrak{R}\left(\exp(\frac{2\pi}{\beta}A)\phi\right)\\ &= \mathfrak{R}\left(\exp(\frac{2\pi}{\beta}(A - i\beta))\phi\right)\\ &= \mathfrak{R}\left(\sum_{k=0}^{\infty}\left(\frac{2\pi}{\beta}\right)^k (A - i\beta)^k\phi\right)\\ &\overset{(H2)}{=} \mathfrak{R}(\phi)\end{aligned}$$

and analogous computations for $\mathfrak{I}(\phi)$. For the reverse inclusion "$\subset$" let $x \in \ker(L)$ and define $v(t) := \exp(\frac{t}{\beta}A)x$. Then v is smooth, 2π-periodic and thus

$$v(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikt} \quad \text{for all } t \in \mathbb{R} \qquad \text{where } c_k \in \mathbb{C}^n.$$

Since v is real-valued, we have $c_k = \overline{c_{-k}}$ for all $k \in \mathbb{Z}$. By definition we have $v'(t) = \frac{1}{\beta}Av(t)$ and thus

$$0 = \sum_{k\in\mathbb{Z}}(ik - \frac{1}{\beta}A)c_k e^{ikt} \quad (t \in \mathbb{R}), \qquad \text{whence } (A - i\beta k)c_k = 0 \text{ for all } k \in \mathbb{Z}.$$

The assumptions (H2),(H3) imply

$$c_k = 0 \quad \text{if } |k| \neq 1, \qquad c_1 \in \mathrm{span}_{\mathbb{C}}\{\phi\}, c_{-1} \in \mathrm{span}_{\mathbb{C}}\{\overline{\phi}\}.$$

So $c_1 = \overline{c_{-1}}$ implies

$$x = v(0) = c_1 + c_{-1} = 2\mathfrak{R}(c_1) \in \mathrm{span}\{\mathfrak{R}(\phi), \mathfrak{I}(\phi)\}.$$

The formula for $\mathrm{ran}(L)$ follows from $\mathrm{ran}(L) = \ker(L^*)^{\perp}$ and

$$\ker(L^*) = \ker\left(\mathrm{id} - \exp(\frac{2\pi}{\beta}A^T)\right) = \mathrm{span}\{\mathfrak{R}(\psi), \mathfrak{I}(\psi)\},$$

which is proved the same way.

Step II: Computing $\ker(F_y(0,\omega_0,\lambda_0))$.
Recall $F_y(0,\omega_0,\lambda_0)[h](t) = h(t) - \exp(\frac{t}{\beta}A)h(0)$ and our aim is to show

$$\ker(F_y(0,\omega_0,\lambda_0)) = \mathrm{span}\{y_1, y_2\}.$$

[32] $\mathfrak{R}(\phi)$ is a vector in $\mathbb{R}^n$ consisting of the real parts of the entries of $\phi \in \mathbb{C}^n$ given by (H2); similar for ψ.
[33] Recall $\exp(A + B) = \exp(A)\exp(B)$ whenever $AB = BA$.

The inclusion "⊃" results from

$$\begin{aligned} F_y(0,\omega_0,\lambda_0)[y_1](t) &= \Re(\phi e^{it}) - \exp(\frac{t}{\beta}A)\Re(\phi) \\ &= \Re(\phi e^{it}) - \Re\left(\exp(\frac{t}{\beta}(A-i\beta))\phi e^{it}\right) \\ &= \Re(\phi e^{it}) - \Re(\phi e^{it}) \\ &= 0 \end{aligned}$$

and similar considerations for y_2. To prove the reverse implication let $h \in \ker(F_y(0,\omega_0,\lambda_0))$. Then

$$\begin{aligned} 0 &= F_y(0,\omega_0,\lambda_0)[h](2\pi) \\ &= h(2\pi) - \exp(\frac{2\pi}{\beta}A)h(0) \\ &= L(h(0)) \end{aligned}$$

where we have used $h(2\pi) = h(0)$ due to $h \in Y$. So $h(0) \in \ker(L)$ and Step I implies $h(0) = a\Re(\phi) + b\Im(\phi)$ for some $a, b \in \mathbb{R}$. Hence, for all $t \in \mathbb{R}$,

$$\begin{aligned} h(t) &= \exp(\frac{t}{\beta}A)h(0) \\ &= a\exp(\frac{t}{\beta}A)\Re(\phi) + b\exp(\frac{t}{\beta}A)\Im(\phi) \\ &= a\Re(\exp(\frac{t}{\beta}(A-i\beta))\phi e^{it}) + b\Im(\exp(\frac{t}{\beta}(A-i\beta))\phi e^{it}) \\ &= a\Re(\phi e^{it}) + b\Im(\phi e^{it}) \\ &= ay_1(t) + by_2(t), \end{aligned}$$

which implies $h \in \operatorname{span}\{y_1, y_2\}$.

Step III: Computing $\operatorname{ran}(F_y(0,\omega_0,\lambda_0))$.
We show[34]

$$\operatorname{ran}(F_y(0,\omega_0,\lambda_0)) = \{z \in Z : \langle \Re(\psi), z(2\pi)\rangle = \langle \Im(\psi), z(2\pi)\rangle = 0\}.$$

We first demonstrate the inclusion "⊂", so let $z := F_y(0,\omega_0,\lambda_0)[h]$ for some $h \in Y$. Then

$$z(2\pi) = h(2\pi) - \exp(\frac{2\pi}{\beta}A)h(0) = L(h(0)) \in \operatorname{ran}(L).$$

So Step I gives the claim. To prove "⊃" assume $z \in Z$ satisfies $z(2\pi) \perp \Re(\psi), \Im(\psi)$. By Step I we know that there is $v \in \mathbb{R}^n$ such that $z(2\pi) = Lv$. Then define $h(t) := \exp(\frac{t}{\beta}A)v + z(t)$, which belongs to Y due to

$$h(2\pi) = \exp(\frac{2\pi}{\beta}A)v + z(2\pi) = v = h(0).$$

[34]Step II and III imply that $F_y(0,\omega_0,\lambda_0)$ is a $(2,2)$-Fredholm operator.

Moreover,

$$F_y(0,\omega_0,\lambda_0)[h](t) = h(t) - \exp(\frac{t}{\beta}A)h(0) = h(t) - \exp(\frac{t}{\beta}A)v = z(t),$$

which is all we had to show.

Step IV: Computing $F_{y\omega}(0,\omega_0,\lambda_0)[y_1](2\pi)$.
For any given $\xi \in Y$ we have the formula

$$F_y(0,\omega,\lambda)[\xi](t) = \xi(t) - e^{\omega t A}\xi(0) - \omega \int_0^t e^{\omega(t-\tau)A}(f_x(\lambda,0) - A)[\xi(\tau)]\,d\tau,$$

and so $F_{y\omega}(0,\omega_0,\lambda_0)[\xi](t) = -tAe^{\omega_0 tA}\xi(0)$ implies

$$\begin{aligned} F_{y\omega}(0,\omega_0,\lambda_0)[y_1](2\pi) &= -2\pi A e^{2\pi\omega_0 A} y_1(0) \;=\; -2\pi A e^{\frac{2\pi}{\beta}A}\Re(\phi) \\ &= -2\pi A \Re\left(e^{\frac{2\pi}{\beta}(A-i\beta)}\phi\right) \\ &= -2\pi A\Re(\phi) \;=\; -2\pi\Re(A\phi) \\ &= -2\pi\Re(i\beta\phi) \;=\; 2\pi\beta\Im(\phi). \end{aligned}$$

Step V: Computing $F_{y\lambda}(0,\omega_0,\lambda_0)[y_1](2\pi)$.
We have the formulas

$$\begin{aligned} F_{y\lambda}(0,\omega_0,\lambda_0)[\xi](t) &= -\omega_0 \int_0^t e^{\omega_0(t-\tau)A} f_{x\lambda}(\lambda_0,0)[\xi(\tau)]\,d\tau, \\ F_{y\lambda}(0,\omega_0,\lambda_0)[y_1](2\pi) &= -\omega_0 \int_0^{2\pi} e^{\omega_0(2\pi-\tau)A} f_{x\lambda}(\lambda_0,0)[y_1(\tau)]\,d\tau \\ &= \Re\left(-\frac{1}{\beta}\int_0^{2\pi} e^{\frac{2\pi-\tau}{\beta}A} f_{x\lambda}(\lambda_0,0)\phi e^{i\tau}\,d\tau\right). \end{aligned}$$

Using Proposition 7.1 (ii) for $\xi := f_{x\lambda}(\lambda_0,0)\phi$ we find for some $w \in \mathbb{C}^n$

$$\begin{aligned} F_{y\lambda}(0,\omega_0,\lambda_0)[\xi](t) &= \Re\left((I - e^{\frac{2\pi}{\beta}A})w - \frac{2\pi}{\beta}\langle\xi,\psi\rangle_{\mathbb{C}^n}\phi\right) \\ &= L(\Re(w)) - \frac{2\pi}{\beta}\Re(\langle\xi,\psi\rangle_{\mathbb{C}^n}\phi). \end{aligned}$$

Step VI: Bijectivity of $\hat{F}_{(v,\omega,\lambda)}(0,0,\omega_0,\lambda_0)$.
We only show surjectivity[35]. Let $z \in Z$. We want to find $v \in V, \omega, \lambda \in \mathbb{R}$ such that the following equivalent statements hold:

$$\hat{F}_{(v,\omega,\lambda)}(0,0,\omega_0,\lambda_0)[(v,\omega,\lambda)] = z$$

[35] Injectivity essentially follows by the definition of V, Step II and the invertibility of the matrix M further below.

$$
\begin{aligned}
\Leftrightarrow\quad & \hat{F}_v(0,0,\omega_0,\lambda_0)[v] + \omega \hat{F}_\omega(0,0,\omega_0,\lambda_0) + \lambda \hat{F}_\lambda(0,0,\omega_0,\lambda_0) = z \\
\Leftrightarrow\quad & F_y(0,\omega_0,\lambda_0)[v] + \omega F_{y\omega}(0,\omega_0,\lambda_0)[y_1] + \lambda F_{y\lambda}(0,\omega_0,\lambda_0)[y_1] = z \\
\Leftrightarrow\quad & F_y(0,\omega_0,\lambda_0)[v] = z - \omega F_{y\omega}(0,\omega_0,\lambda_0)[y_1] - \lambda F_{y\lambda}(0,\omega_0,\lambda_0)[y_1]
\end{aligned}
$$

A solution exists if and only if the right hand side lies in $\operatorname{ran}(F_y(0,\omega_0,\lambda_0))$. From Step IV we know that this holds if and only if the right hand side evaluated at $t = 2\pi$ is orthogonal to $\Re(\psi), \Im(\psi)$. So we have to find $\omega, \lambda \in \mathbb{R}$ such that

$$
\begin{aligned}
\langle (z - \omega F_{y\omega}(0,\omega_0,\lambda_0)[y_1] - \lambda F_{y\lambda}(0,\omega_0,\lambda_0)[y_1])(2\pi), \Re(\psi)\rangle = 0, \\
\langle (z - \omega F_{y\omega}(0,\omega_0,\lambda_0)[y_1] - \lambda F_{y\lambda}(0,\omega_0,\lambda_0)[y_1])(2\pi), \Im(\psi)\rangle = 0.
\end{aligned}
$$

This is a linear 2×2-system that can be rewritten as

$$
M \begin{pmatrix} \omega \\ \lambda \end{pmatrix} = \begin{pmatrix} \langle z(2\pi), \Re(\psi)\rangle \\ \langle z(2\pi), \Im(\psi)\rangle \end{pmatrix}
$$

where

$$
M := \begin{pmatrix} \langle F_{y\omega}(0,\omega_0,\lambda_0)[y_1](2\pi), \Re(\psi)\rangle & \langle F_{y\lambda}(0,\omega_0,\lambda_0)[y_1](2\pi), \Re(\psi)\rangle \\ \langle F_{y\omega}(0,\omega_0,\lambda_0)[y_1](2\pi), \Im(\psi)\rangle & \langle F_{y\lambda}(0,\omega_0,\lambda_0)[y_1](2\pi), \Im(\psi)\rangle \end{pmatrix}.
$$

The steps IV and V allow to compute the entries of this matrix[36] ($\xi := f_{x\lambda}(\lambda_0,0)\phi$):

$$
\begin{aligned}
\langle F_{y\omega}(0,\omega_0,\lambda_0)[y_1](2\pi), \Re(\psi)\rangle &= 2\pi\beta\langle \Im(\phi), \Re(\psi)\rangle = 0, \\
\langle F_{y\omega}(0,\omega_0,\lambda_0)[y_1](2\pi), \Im(\psi)\rangle &= 2\pi\beta\langle \Im(\phi), \Im(\psi)\rangle = \pi\beta, \\
\langle F_{y\lambda}(0,\omega_0,\lambda_0)[y_1](2\pi), \Re(\psi)\rangle &= \langle L(\Re(w)) - \frac{2\pi}{\beta}\Re(\langle \xi,\psi\rangle_{\mathbb{C}^n}\phi), \Re(\psi)\rangle \\
&= -\frac{2\pi}{\beta}\langle \Re(\langle \xi,\psi\rangle_{\mathbb{C}^n}\phi), \Re(\psi)\rangle \\
&= -\frac{2\pi}{\beta}\langle \Re(\langle \xi,\psi\rangle_{\mathbb{C}^n})\Re(\phi) - \Im(\langle \xi,\psi\rangle_{\mathbb{C}^n})\Im(\phi), \Re(\psi)\rangle \qquad (7.5) \\
&= -\frac{\pi}{\beta}\Re(\langle \xi,\psi\rangle_{\mathbb{C}^n}), \\
\langle F_{y\lambda}(0,\omega_0,\lambda_0)[y_1](2\pi), \Im(\psi)\rangle &= -\frac{2\pi}{\beta}\langle \Re(\langle \xi,\psi\rangle_{\mathbb{C}^n}\phi), \Im(\psi)\rangle \\
&= \dots \\
&= \frac{\pi}{\beta}\Im(\langle \xi,\psi\rangle_{\mathbb{C}^n}).
\end{aligned}
$$

We conclude

$$
\det(M) = 0 \cdot \frac{\pi}{\beta}\Im(\langle \psi,\xi\rangle_{\mathbb{C}^n}) - \pi\beta \cdot \left(-\frac{\pi}{\beta}\Re(\langle \psi,\xi\rangle_{\mathbb{C}^n})\right) = \pi^2 \Re(\langle \psi, f_{x\lambda}(\lambda_0,0)\phi\rangle_{\mathbb{C}^n}) \neq 0
$$

[36] Recall $\phi \cdot \bar{\psi} = 1, \phi \cdot \psi = 0$ and thus

$$
\langle \Im(\phi), \Re(\psi)\rangle = 0, \quad \langle \Im(\phi), \Im(\psi)\rangle = \frac{1}{2}, \quad \langle \Re(\phi), \Re(\psi)\rangle = \frac{1}{2}, \quad \langle \Re(\phi), \Im(\psi)\rangle = 0. \qquad (7.4)
$$

by the transversality condition (H4). So the above linear system is (uniquely) solvable and the claim is proved.

End of Lec15

Step VII: Uniqueness up to translations.
Let (y_n^*) be a sequence of nontrivial T_n-periodic solutions of (7.1) for $\lambda = \lambda_n$ such that

$$\|y_n^*\|_{C([0,T_n])} + |\lambda_n - \lambda_0| + |T_n - 2\pi\omega_0| \to 0 \quad \text{as } n \to \infty.$$

Then the functions $\tilde{y}_n(t) := y_n^*(\omega_n t)$ with $\omega_n := \frac{T_n}{2\pi}$ are 2π-periodic solutions of the rescaled ODE (7.2) for $(\omega, \lambda) = (\omega_n, \lambda_n)$ and

$$\|\tilde{y}_n\|_{C([0,2\pi])} + |\lambda_n - \lambda_0| + |\omega_n - \omega_0| \to 0 \quad \text{as } n \to \infty.$$

Since the projectors mapping Y onto the subspaces $\operatorname{span}\{y_1, y_2\}$ and V are bounded, we find a null sequence (s_n) and $\theta_n \in [0, 2\pi)$ such that

$$\tilde{y}_n = s_n \cos(\theta_n) y_1 - s_n \sin(\theta_n) y_2 + v_n$$

with $v_n \to 0$ in V as $n \to \infty$. We use without proof[37] that necessarily $s_n^{-1} v_n \to 0$ as $n \to \infty$ holds. Then, for all $t \in \mathbb{R}$,

$$\begin{aligned}
\tilde{y}_n(t - \theta_n) &= s_n \cos(\theta) y_1(t - \theta_n) - \sin(\theta_n) y_2(t - \theta_n) + v_n(t - \theta_n) \\
&\overset{(7.3)}{=} s_n \cos(\theta_n) \mathfrak{R}(e^{i(t-\theta_n)}\phi) - \sin(\theta_n) \mathfrak{I}(e^{i(t-\theta_n)}\phi) + v_n(t - \theta_n) \\
&= s_n \mathfrak{R}(e^{i\theta_n} \cdot e^{i(t-\theta_n)}\phi) + v_n(t - \theta_n) \\
&= s_n \mathfrak{R}(e^{it}\phi) + v_n(t - \theta_n) \\
&= s_n y_1(t) + \underbrace{v_n(t - \theta_n)}_{\in V}.
\end{aligned}$$

Here we use that V is a translation-invariant subspace of Y. As a solution of (7.1) the function $\tilde{y}_n(\cdot - \theta_n)$ is a nontrivial solution of $F(y, \omega_n, \lambda_n) = 0$ and hence $\hat{F}(s_n, \frac{v_n(\cdot - \theta_n)}{s_n}, \omega_n, \lambda_n) = 0$ for almost all $n \in \mathbb{N}$. From $\frac{v_n(\cdot - \theta_n)}{s_n} \to 0$ in V and the uniqueness property provided by the Implicit Function Theorem we get $v_n(t - \theta_n) = s_n \hat{v}(s_n)(t), \omega_n = \hat{\omega}(s_n), \lambda_n = \hat{\lambda}(s_n)$ for almost all $n \in \mathbb{N}$. In particular, we get for almost all $n \in \mathbb{N}$

$$y_n^*(t) = \tilde{y}_n(\omega_n^{-1} t) = s_n y_1(\omega_n^{-1} t + \theta_n) + s_n \hat{v}(s_n)(\omega_n^{-1} t + \theta_n) = \hat{y}(s_n)(\omega_n^{-1} t + \theta) \qquad \text{for all } t \in \mathbb{R}.$$

In view of $\omega_n^{-1} = 2\pi T_n^{-1}$ we have thus shown that almost all solutions are given by

$$y_n^* = \hat{y}(s_n)(2\pi T_n^{-1}(\cdot + \theta_n)), \qquad \omega_n = \hat{\omega}(s_n), \quad \lambda_n = \hat{\lambda}(s_n),$$

which is all we had to show. □

[37] Given that $F_y(0, \omega_0, \lambda_0)$ is a $(2,2)$-Fredholm operator this can be deduced from $\hat{y}(\lambda_0, 0) = 0, \hat{y}_\psi(\lambda_0, 0) = 0$ in the context of (6.2).

The following is not relevant for the oral exam!

As before it is possible to determine the direction of bifurcation. To this end we compute the relevant derivatives of $\hat{F}$ at the point $(0,0,\omega_0,\lambda_0)$. In view of

$$\hat{F}(s,v,\omega,\lambda) = \int_0^1 F_y(s\tau(y_1+v),\omega,\lambda)[y_1+v]\,d\tau$$

we get after some computations for $h \in V$

$$\begin{aligned} \hat{F}_s(0,0,\omega_0,\lambda_0) &= \frac{1}{2}F_{yy}(0,\omega_0,\lambda_0)[y_1,y_1], \\ \hat{F}_v(0,0,\omega_0,\lambda_0)[h] &= F_y(0,\omega_0,\lambda_0)[h], \\ \hat{F}_\omega(0,0,\omega_0,\lambda_0) &= F_{y\omega}(0,\omega_0,\lambda_0)[y_1], \\ \hat{F}_\lambda(0,0,\omega_0,\lambda_0) &= F_{y\lambda}(0,\omega_0,\lambda_0)[y_1], \\ \hat{F}_{ss}(0,0,\omega_0,\lambda_0) &= \frac{1}{3}F_{yyy}(0,\omega_0,\lambda_0)[y_1,y_1,y_1], \\ \hat{F}_{sv}(0,0,\omega_0,\lambda_0)[h] &= F_{yy}(0,\omega_0,\lambda_0)[y_1,h]. \end{aligned} \tag{7.6}$$

Corollary 7.3. *Under the assumptions of the theorem we have* $\hat{y}'(0) = y_1$ *as well as* $\hat{\lambda}'(0) = \hat{\omega}'(0) = 0$. *If* $f \in C^3(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$ *then*

$$\hat{\lambda}''(0) = -\frac{\Re(\langle \xi^*,\psi\rangle_{\mathbb{C}^n})}{\Re(\langle \xi,\psi\rangle_{\mathbb{C}^n})}, \qquad \hat{\omega}''(0) = \frac{\Im\left(\langle \xi,\psi\rangle_{\mathbb{C}^n}\overline{\langle \xi^*,\psi\rangle_{\mathbb{C}^n}}\right)}{\beta^2\Re(\langle \xi,\psi\rangle_{\mathbb{C}^n})}$$

where $\xi := f_{x\lambda}(\lambda_0,0)[\phi]$ *and*

$$\begin{aligned} \xi^* := &\frac{1}{4}f_{yyy}(\lambda_0,0)[\phi,\phi,\bar{\phi}] - \frac{1}{2}f_{yy}(\lambda_0,0)[\phi,A^{-1}(f_{xx}(\lambda_0,0)[\phi,\bar{\phi}])] \\ &- \frac{1}{4}f_{yy}(\lambda_0,0)[\bar{\phi},(A-2\beta i)^{-1}(f_{xx}(\lambda_0,0)[\phi,\phi])]. \end{aligned}$$

Proof. We use $\hat{F}(s,\hat{v}(s),\hat{\omega}(s),\hat{\lambda}(s)) = 0$ for all $|s| < \varepsilon$ and differentiate this identity. This gives for $v = \hat{v}(s), \omega = \hat{\omega}(s), \lambda = \hat{\lambda}(s)$

$$0 = \hat{F}_s(s,v,\omega,\lambda) + \hat{F}_v(s,v,\omega,\lambda)[\hat{v}'(s)] + \hat{F}_\omega(s,v,\omega,\lambda)\hat{\omega}'(s) + \hat{F}_\lambda(s,v,\omega,\lambda)\hat{\lambda}'(s).$$

We first plug in $s = 0$ and obtain in view of (7.6)

$$\begin{aligned} 0 = &\frac{1}{2}F_{yy}(0,\omega_0,\lambda_0)[y_1,y_1] + F_y(0,\omega_0,\lambda_0)[\hat{v}'(0)] \\ &+ F_{y\omega}(0,\omega_0,\lambda_0)[y_1]\hat{\omega}'(0) + F_{y\lambda}(0,\omega_0,\lambda_0)[y_1]\hat{\lambda}'(0). \end{aligned} \tag{7.7}$$

We want to retrieve $\hat{v}'(0), \hat{\omega}'(0), \hat{\lambda}'(0)$. To this end we compute

$$\begin{aligned}
F_{yy}(0,\omega_0,\lambda_0)[y_1,y_1](t) &= -\omega_0 \int_0^t e^{\omega_0(t-\tau)A} f_{yy}(\lambda_0,0)[y_1(\tau),y_1(\tau)]\,d\tau \\
&= -\frac{\omega_0}{4}\int_0^t e^{\omega_0(t-\tau)A} f_{yy}(\lambda_0,0)[e^{i\tau}\phi + e^{-i\tau}\bar{\phi}, e^{i\tau}\phi + e^{-i\tau}\bar{\phi}]\,d\tau \\
&= -\frac{\omega_0}{4}\int_0^t e^{\omega_0(t-\tau)A} f_{yy}(\lambda_0,0)[\phi,\phi]e^{2i\tau}\,d\tau \\
&\quad -\frac{\omega_0}{4}\int_0^t e^{\omega_0(t-\tau)A} f_{yy}(\lambda_0,0)[\bar{\phi},\bar{\phi}]e^{-2i\tau}\,d\tau \\
&\quad -\frac{\omega_0}{2}\int_0^t e^{\omega_0(t-\tau)A} f_{yy}(\lambda_0,0)[\phi,\bar{\phi}]\,d\tau.
\end{aligned}$$

So Proposition 7.1 (i) gives (recall $\omega_0 = \frac{1}{\beta}$)

$$\begin{aligned}
-\frac{1}{2}F_{yy}(0,\omega_0,\lambda_0)[y_1,y_1](t) &= -\frac{1}{8}(e^{2it} - e^{\omega_0 tA})(A-2i\beta)^{-1} f_{yy}(\lambda_0,0)[\phi,\phi] \\
&\quad -\frac{1}{8}(e^{-2it} - e^{\omega_0 tA})(A+2i\beta)^{-1} f_{yy}(\lambda_0,0)[\bar{\phi},\bar{\phi}] \\
&\quad -\frac{1}{4}(I - e^{\omega_0 tA})A^{-1} f_{yy}(\lambda_0,0)[\phi,\bar{\phi}] \\
&= \underbrace{v_0(t) - e^{\omega_0 tA}v_0(0)}_{=F_y(0,\omega_0,\lambda_0)[v_0](t)}
\end{aligned}$$

for

$$v_0(t) = \xi_1 + \xi_2 e^{2it} + \bar{\xi}_2 e^{-2it} \qquad \text{and}$$

$$\xi_1 := -\frac{1}{4}A^{-1}(f_{xx}(\lambda_0,0)[\phi,\bar{\phi}]) \in \mathbb{R}^n, \qquad \xi_2 := -\frac{1}{8}(A-2\beta i)^{-1}(f_{xx}(\lambda_0,0)[\phi,\phi]) \in \mathbb{C}^n.$$

Given that (7.7) is uniquely solvable (see Step VI), we find

$$\hat{v}'(0) = v_0, \qquad \hat{\omega}'(0) = \hat{\lambda}'(0) = 0.$$

We now want to determine $\hat{\omega}''(0), \hat{\lambda}''(0)$. To do this we differentiate the equation once more and evaluate the resulting expression at $s = 0$. Due to $\hat{\omega}'(0) = \hat{\lambda}'(0) = 0$ this complicated expression simplifies to

$$\begin{aligned}
0 &= \hat{F}_{ss}(0,0,\omega_0,\lambda_0) + 2\hat{F}_{sv}(0,0,\omega_0,\lambda_0)[\hat{v}'(0)] + \hat{F}_v(0,0,\omega_0,\lambda_0)[\hat{v}''(0)] \\
&\quad + \hat{F}_{vv}(0,0,\omega_0,\lambda_0)[\hat{v}'(0),\hat{v}'(0)] + \hat{F}_\omega(0,0,\omega_0,\lambda_0)\hat{\omega}''(0) + \hat{F}_\lambda(0,0,\omega_0,\lambda_0)\hat{\lambda}''(0) \\
&= \frac{1}{3}F_{yyy}(0,\omega_0,\lambda_0)[y_1,y_1,y_1] + 2F_{yy}(0,\omega_0,\lambda_0)[y_1,v_0] + F_y(0,\omega_0,\lambda_0)[\hat{v}''(0)] \\
&\quad + F_{y\omega}(0,\omega_0,\lambda_0)[y_1]\hat{\omega}''(0) + F_{y\lambda}(0,\omega_0,\lambda_0)[y_1]\hat{\lambda}''(0).
\end{aligned}$$

We are not interested in $\hat{v}''(0)$, so we "project it out". This means that we evaluate this identity at $t = 2\pi$ and test the resulting equation with $\mathfrak{R}(\psi), \mathfrak{I}(\psi)$. Step III implies

$$M \begin{pmatrix} \hat{\omega}''(0) \\ \hat{\lambda}''(0) \end{pmatrix} = \begin{pmatrix} \langle -\frac{1}{3} F_{yyy}(0,\omega_0,\lambda_0)[y_1,y_1,y_1](2\pi) - 2F_{yy}(0,\omega_0,\lambda_0)[y_1,v_0](2\pi), \mathfrak{R}(\psi)\rangle \\ \langle -\frac{1}{3} F_{yyy}(0,\omega_0,\lambda_0)[y_1,y_1,y_1](2\pi) - 2F_{yy}(0,\omega_0,\lambda_0)[y_1,v_0](2\pi), \mathfrak{I}(\psi)\rangle \end{pmatrix}.$$

We compute:

$$\begin{aligned}
&-\frac{1}{3} F_{yyy}(0,\omega_0,\lambda_0)[y_1,y_1,y_1](2\pi) \\
&= -\frac{1}{3}\left(-\omega_0 \int_0^{2\pi} e^{\omega_0(2\pi-\tau)A} f_{yyy}(\lambda_0,0)[y_1(\tau),y_1(\tau),y_1(\tau)]\, d\tau\right) \\
&= -\frac{1}{24}\left(-\omega_0 \int_0^{2\pi} e^{\omega_0(2\pi-\tau)A} f_{yyy}(\lambda_0,0)[\phi e^{i\tau} + \bar{\phi} e^{-i\tau}, \phi e^{i\tau} + \bar{\phi} e^{-i\tau}, \phi e^{i\tau} + \bar{\phi} e^{-i\tau}]\, d\tau\right) \\
&= -\frac{1}{12}\mathfrak{R}\left(-\omega_0 \int_0^{2\pi} e^{\omega_0(2\pi-\tau)A} f_{yyy}(\lambda_0,0)[\phi,\phi,\phi] e^{3i\tau}\, d\tau\right) \\
&\quad -\frac{1}{4}\mathfrak{R}\left(-\omega_0 \int_0^{2\pi} e^{\omega_0(2\pi-\tau)A} f_{yyy}(\lambda_0,0)[\phi,\phi,\bar{\phi}] e^{i\tau}\, d\tau\right).
\end{aligned}$$

These integrals can be evaluated with the aid of Proposition 7.1, which gives

$$-\frac{1}{3} F_{yyy}(0,\omega_0,\lambda_0)[y_1,y_1,y_1](2\pi) \in \operatorname{ran}(L) + \frac{\pi}{2\beta}\mathfrak{R}(\langle \xi_3, \psi\rangle_{\mathbb{C}^n} \phi)$$

for $\xi_3 := f_{yyy}(\lambda_0,0)[\phi,\phi,\bar{\phi}]$. The other term is treated as follows:

$$\begin{aligned}
&-2F_{yy}(0,\omega_0,\lambda_0)[y_1,v_0](2\pi) \\
&= -2\left(-\omega_0 \int_0^{2\pi} e^{\omega_0(2\pi-\tau)A} f_{yy}(\lambda_0,0)[y_1(\tau),v_0(\tau)]\, d\tau\right) \\
&= -\left(-\omega_0 \int_0^{2\pi} e^{\omega_0(2\pi-\tau)A} f_{yy}(\lambda_0,0)[\phi e^{i\tau} + \bar{\phi} e^{-i\tau}, \xi_1 + \xi_2 e^{2i\tau} + \bar{\xi}_2 e^{-2i\tau}]\, d\tau\right) \\
&\in \operatorname{ran}(L) - 2\mathfrak{R}\left(-\omega_0 \int_0^{2\pi} e^{\omega_0(2\pi-\tau)A} (f_{yy}(\lambda_0,0)[\phi,\xi_1] + f_{yy}(\lambda_0,0)[\bar{\phi},\xi_2]) e^{i\tau}\, d\tau\right) \\
&\in \operatorname{ran}(L) - 2\mathfrak{R}\left(-\frac{2\pi}{\beta}\langle \xi_4,\psi\rangle_{\mathbb{C}^n}\phi\right) \\
&\in \operatorname{ran}(L) + \frac{4\pi}{\beta}\mathfrak{R}\left(\langle \xi_4,\psi\rangle_{\mathbb{C}^n}\phi\right)
\end{aligned}$$

for $\xi_4 := f_{yy}(\lambda_0,0)[\phi,\xi_1] + f_{yy}(\lambda_0,0)[\bar{\phi},\xi_2]$. Hence, for $\xi_5 := \frac{1}{4}\xi_3 + 2\xi_4$,

$$M \begin{pmatrix} \hat{\omega}''(0) \\ \hat{\lambda}''(0) \end{pmatrix} = \begin{pmatrix} \langle \frac{\pi}{2\beta}\mathfrak{R}(\langle \xi_3,\psi\rangle_{\mathbb{C}^n}\phi) + \frac{4\pi}{\beta}\mathfrak{R}\left(\langle \xi_4,\psi\rangle_{\mathbb{C}^n}\phi\right), \mathfrak{R}(\psi)\rangle \\ \langle \frac{\pi}{2\beta}\mathfrak{R}(\langle \xi_3,\psi\rangle_{\mathbb{C}^n}\phi) + \frac{4\pi}{\beta}\mathfrak{R}\left(\langle \xi_4,\psi\rangle_{\mathbb{C}^n}\phi\right), \mathfrak{I}(\psi)\rangle \end{pmatrix} \overset{(7.4)}{=} \frac{\pi}{\beta}\begin{pmatrix} \mathfrak{R}(\langle \xi_5,\psi\rangle_{\mathbb{C}^n}) \\ -\mathfrak{I}(\langle \xi_5,\psi\rangle_{\mathbb{C}^n}) \end{pmatrix}.$$

From (7.5) we infer

$$\begin{pmatrix} \hat{\omega}''(0) \\ \hat{\lambda}''(0) \end{pmatrix} = \frac{\pi}{\beta} M^{-1} \begin{pmatrix} \mathfrak{R}(\langle \xi_5,\psi\rangle_{\mathbb{C}^n}) \\ -\mathfrak{I}(\langle \xi_5,\psi\rangle_{\mathbb{C}^n}) \end{pmatrix}$$

$$
\begin{aligned}
&= \frac{\pi}{\beta}\begin{pmatrix} \frac{1}{\pi\beta}\frac{\Im(\langle\xi,\psi\rangle_{\mathbb{C}^n})}{\Re(\langle\xi,\psi\rangle_{\mathbb{C}^n})} & \frac{1}{\pi\beta} \\ -\frac{\beta}{\pi\Re(\langle\xi,\psi\rangle_{\mathbb{C}^n})} & 0 \end{pmatrix}\begin{pmatrix} \Re(\langle\xi_5,\psi\rangle_{\mathbb{C}^n}) \\ -\Im(\langle\xi_5,\psi\rangle_{\mathbb{C}^n}) \end{pmatrix} \\
&= \frac{1}{\beta^2\Re(\langle\xi,\psi\rangle_{\mathbb{C}^n})}\begin{pmatrix} \Im(\langle\xi,\psi\rangle_{\mathbb{C}^n})\Re(\langle\xi_5,\psi\rangle_{\mathbb{C}^n}) - \Im(\langle\xi_5,\psi\rangle_{\mathbb{C}^n})\Re(\langle\xi,\psi\rangle_{\mathbb{C}^n}) \\ -\beta^2\Re(\langle\xi_5,\psi\rangle_{\mathbb{C}^n}) \end{pmatrix} \\
&= \frac{1}{\beta^2\Re(\langle\xi,\psi\rangle_{\mathbb{C}^n})}\begin{pmatrix} \Im\left(\langle\xi,\psi\rangle_{\mathbb{C}^n}\overline{\langle\xi_5,\psi\rangle_{\mathbb{C}^n})}\right) \\ -\beta^2\Re(\langle\xi_5,\psi\rangle_{\mathbb{C}^n}) \end{pmatrix}.
\end{aligned}
$$

So the claim follows from

$$
\begin{aligned}
\xi_5 &= \frac{1}{4}\xi_3 + 2\xi_4 \\
&= \frac{1}{4}f_{yyy}(\lambda_0,0)[\phi,\phi,\bar{\phi}] + 2f_{yy}(\lambda_0,0)[\phi,\xi_1] + 2f_{yy}(\lambda_0,0)[\bar{\phi},\xi_2] \\
&= \xi^*
\end{aligned}
$$

□

Example 7.4 (Van der Pol oscillator)**.** Consider the ODE

$$u'' - \lambda(1-u^2)u' + u = 0.$$

Then $y := (u,u')$ satisfies $y' = f(\lambda,y)$ where

$$f(\lambda,y) = \begin{pmatrix} y_2 \\ -y_1 + \lambda(1-y_1^2)y_2 \end{pmatrix}.$$

This function is smooth, satisfies $f(\lambda,0) = 0$ for all $\lambda \in \mathbb{R}$, so (H1) is satisfied. Due to

$$f_y(\lambda,0) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}$$

the assumptions (H2),(H3) hold for $\lambda_0 = 0, \beta = 1, \psi = \phi = \frac{1}{\sqrt{2}}(1,i)^T$. Finally, (H4) holds because

$$\Re(\langle f_{y\lambda}(\lambda_0,0)[\phi],\psi\rangle_{\mathbb{C}^2}) = \frac{1}{2}\Re\left(\langle\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 \\ i\end{pmatrix},\begin{pmatrix} 1 \\ i\end{pmatrix}\rangle_{\mathbb{C}^2}\right) = \frac{1}{2}\Re\left(\langle\begin{pmatrix} 0 \\ i\end{pmatrix},\begin{pmatrix} 1 \\ i\end{pmatrix}\rangle_{\mathbb{C}^2}\right) = \frac{1}{2} \neq 0.$$

So Hopf bifurcation occurs at $\lambda = \lambda_0 = 0$ and the solutions have periodicities close to $\hat{T}(0) = \frac{2\pi}{\beta} = 2\pi$. The bifurcation direction can be computed as well. In view of $f_{yy}(\lambda_0,0) = 0, f_{yyy}(\lambda_0,0) = 0$ we have $\xi^* = 0$ and thus $\hat{\lambda}'(0) = \hat{T}'(0) = \hat{\lambda}''(0) = \hat{T}''(0) = 0$. This is no surprise given that the bifurcating solutions are given by

$$\hat{\lambda}(s) = \lambda,\ \hat{T}(s) = 2\pi,\ \hat{x}(s)(t) = const(s) * \sin(t)$$

For the slightly different model

$$u'' - (\lambda - u^2)u' + u = 0$$

essentially the same computations reveal that Hopf bifurcation occurs (same $\phi, \psi, \beta, \lambda_0$ as above), but now $f(\lambda, y) = (y_2, -y_1 + \lambda y_2 - y_1^2 y_2)^T$ gives

$$f_{yy}(\lambda_0, 0) = 0, \qquad f_{yyy}(\lambda_0, 0)[\eta^1, \eta^2, \eta^3] = (0, -2\eta_2^1\eta_1^2\eta_1^3 - 2\eta_1^1\eta_2^2\eta_1^3 - 2\eta_1^1\eta_1^2\eta_2^3)$$

and

$$\xi^* = \frac{1}{4} f_{yyy}(\lambda_0, 0)[\phi, \phi, \bar{\phi}] = \frac{1}{4}\begin{pmatrix} 0 \\ -4|\phi_1|^2\phi_2 - 2\phi_1^2\bar{\phi}_2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 \\ -4 \cdot \frac{i}{2\sqrt{2}} - 2 \cdot \frac{-i}{2\sqrt{2}} \end{pmatrix} = -\frac{i}{4\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $\xi := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \end{pmatrix}$ implies

$$\langle \xi^*, \psi \rangle_{\mathbb{C}^n} = -\frac{i}{8}\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix} \rangle_{\mathbb{C}^n} = -\frac{1}{8}, \qquad \langle \xi, \psi \rangle_{\mathbb{C}^n} = \frac{1}{2}$$

and thus

$$\hat{\lambda}''(0) = -\frac{\Re(\langle \xi^*, \psi \rangle_{\mathbb{C}^n})}{\Re(\langle \xi, \psi \rangle_{\mathbb{C}^n})} = \frac{1}{4}, \qquad \hat{\omega}''(0) = \frac{\Im\left(\langle \xi, \psi \rangle_{\mathbb{C}^n}\overline{\langle \xi^*, \psi \rangle_{\mathbb{C}^n}}\right)}{\beta^2 \Re(\langle \xi, \psi \rangle_{\mathbb{C}^n})} = 0.$$

End of Lec16

References

[1] A. Ambrosetti and G. Prodi. *A primer of nonlinear analysis*, volume 34 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Corrected reprint of the 1993 original.

[2] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[3] K.-C. Chang. *Methods in nonlinear analysis*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[4] M. G. Crandall and P. H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Functional Analysis*, 8:321–340, 1971. doi:10.1016/0022-1236(71)90015-2.

[5] M. G. Crandall and P. H. Rabinowitz. The Hopf bifurcation theorem in infinite dimensions. *Arch. Rational Mech. Anal.*, 67(1):53–72, 1977. doi:10.1007/BF00280827.

[6] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010. doi:10.1090/gsm/019.

[7] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[8] E. Hopf. Abzweigung einer periodischen Lösung von einer stationären eines Differentialsystems. *Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Nat. Kl.*, 95(1):3–22, 1943.

[9] H. Kielhöfer. *Bifurcation theory*, volume 156 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004. An introduction with applications to PDEs. doi:10.1007/b97365.

[10] M. G. Kreĭn and M. A. Rutman. Linear operators leaving invariant a cone in a Banach space. *Amer. Math. Soc. Translation*, 1950(26):128, 1950.

[11] P. H. Rabinowitz. On bifurcation from infinity. *J. Differential Equations*, 14:462–475, 1973. doi:10.1016/0022-0396(73)90061-2.