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Problem Sheet 10

Bifurcation Theory

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Problem 25 (Uniqueness of decomposition in Theorem 5.2):
Let X, Z be real Banach spaces, $\lambda_0 \in \mathbb{R}$, $F\colon \mathbb{R} \times X \to Z$ with

$$F(\lambda, x) = L(\lambda)x + R(\lambda, x) = \tilde{L}(\lambda)x + \tilde{R}(\lambda, x)$$

for $(\lambda, x) \in \mathbb{R} \times X$, where (L, R) satisfy

(A) $L \in C^1(\mathbb{R}, \mathcal{L}(X; Z))$, $R \in C^1(\mathbb{R} \times X; Z)$.

(S) $L(\lambda_0)$ is a $(1,1)$-Fredholm operator, and let $\phi \in X \setminus \{0\}$ such that $\mathbb{R}\phi = \ker(L(\lambda_0))$.

(R) There is an open neighbourhood $\mathcal{U} \subseteq \mathbb{R} \times X \times \mathbb{R}$ of $(\lambda_0, \phi, 0)$ such that $(\lambda, x, s) \mapsto sR(\lambda, \frac{1}{s}x)$ has an extension $\mathcal{R} \in C^1(\mathcal{U}; Z)$ with

$$\mathcal{R}(\lambda_0, \phi, 0) = 0, \quad \mathcal{R}_\lambda(\lambda_0, \phi, 0) = 0, \quad \mathcal{R}_x(\lambda_0, \phi, 0) = 0.$$

Assume $(\tilde{L}, \tilde{R})$ satisfy the same assumptions and $\ker(L(\lambda_0)) = \ker(\tilde{L}(\lambda_0))$. Show that $L(\lambda_0) = \tilde{L}(\lambda_0)$ and $R(\lambda_0, \cdot\,) = \tilde{R}(\lambda_0, \cdot\,)$.

Problem 26:
Similar to Example 5.4 in the lecture, consider

$$\begin{cases} -u'' - \lambda u = \frac{a(x)u}{1+b(x)u^2} \text{ in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

where $a, b \in C([0,1])$ and $b > 0$. Assume in addition that $a(x) = 0$ whenever x is near 0 or near 1. Show that bifurcation from ∞ occurs at $\lambda_0 = \pi^2$.