Institute for Analysis
PD Dr. Rainer Mandel
M.Sc. Sebastian Ohrem

Solution to Problem Sheet 10

Bifurcation Theory

Winter Semester 2022/23

23.1.2023

Problem 25 (Uniqueness of decomposition in Theorem 5.2):
Let X, Z be real Banach spaces, $\lambda_0 \in \mathbb{R}$, $F : \mathbb{R} \times X \to Z$ with

$$F(\lambda, x) = L(\lambda)x + R(\lambda, x) = \tilde{L}(\lambda)x + \tilde{R}(\lambda, x)$$

for $(\lambda, x) \in \mathbb{R} \times X$, where (L, R) satisfy

(A) $L \in C^1(\mathbb{R}, \mathcal{L}(X; Z))$, $R \in C^1(\mathbb{R} \times X; Z)$.

(S) $L(\lambda_0)$ is a $(1,1)$-Fredholm operator, and let $\phi \in X \setminus \{0\}$ such that $\mathbb{R}\phi = \ker(L(\lambda_0))$.

(R) There is an open neighbourhood $\mathcal{U} \subseteq \mathbb{R} \times X \times \mathbb{R}$ of $(\lambda_0, \phi, 0)$ such that $(\lambda, x, s) \mapsto sR(\lambda, \frac{1}{s}x)$ has an extension $\mathcal{R} \in C^1(\mathcal{U}; Z)$ with

$$\mathcal{R}(\lambda_0, \phi, 0) = 0, \quad \mathcal{R}_\lambda(\lambda_0, \phi, 0) = 0, \quad \mathcal{R}_x(\lambda_0, \phi, 0) = 0.$$

Assume $(\tilde{L}, \tilde{R})$ satisfy the same assumptions and $\ker(L(\lambda_0)) = \ker(\tilde{L}(\lambda_0))$. Show that $L(\lambda_0) = \tilde{L}(\lambda_0)$ and $R(\lambda_0, \cdot\,) = \tilde{R}(\lambda_0, \cdot\,)$.

Solution to problem 25:
Proof: First, by assumption (R) there exists a continuous extension $\mathcal{R} : \mathcal{U} \to Z$ of $(\lambda, x, s) \mapsto sR(\lambda, \frac{1}{s}x)$, with $\mathcal{U}$ being a neighbourhood of $(\lambda_0, \phi, 0)$ and $\mathbb{R}\phi = \ker(L(\lambda_0))$.

The same way, we find $\tilde{\mathcal{R}} : \tilde{\mathcal{U}} \to Z$, $(\lambda_0, \tilde{\phi}, 0) \in \tilde{\mathcal{U}}$, $\mathbb{R}\tilde{\phi} = \ker(\tilde{L}(\lambda_0))$.

By assumption, $\tilde{\phi} = c\phi$ for some $c \in \mathbb{R} \setminus \{0\}$.

We now define

$$Q(\lambda, x) := R(\lambda, x) - \tilde{R}(\lambda, x) = \tilde{L}(\lambda)x - L(\lambda)x$$

and consider

$$sQ(\lambda, \tfrac{1}{s}x) = sR(\lambda, \tfrac{1}{s}x) - \tfrac{1}{c}cs\tilde{R}(\lambda, \tfrac{1}{cs}cx) = \mathcal{R}(\lambda, x, s) - \tfrac{1}{c}\tilde{\mathcal{R}}(\lambda, cx, cs),$$

which thus has a continuous extension to

$$U := \mathcal{U} \cap \{(\lambda, x, s) \in \mathbb{R} \times X \times \mathbb{R} : (\lambda, cx, cs) \in \tilde{\mathcal{U}}\},$$

which is a neighbourhood of $(\lambda_0, \phi, 0)$. By (R) we further have

$$\mathcal{Q}(\lambda_0, \phi, 0) = 0, \quad \mathcal{Q}_\lambda(\lambda_0, \phi, 0) = 0, \quad \mathcal{Q}_x(\lambda_0, \phi, 0) = 0.$$

However,

$$\mathcal{Q}(\lambda, x, s) = L(\lambda)x - \tilde{L}(\lambda)x,$$

hence

$$\mathcal{Q}_x(\lambda_0, \phi, 0) = L(\lambda_0) - \tilde{L}(\lambda_0).$$

We conclude $L(\lambda_0) = \tilde{L}(\lambda_0)$ and therefore also $R(\lambda_0, \cdot\,) = \tilde{R}(\lambda_0, \cdot\,)$. □

Problem 26:
Similar to Example 5.4 in the lecture, consider

$$\begin{cases} -u'' - \lambda u = \frac{a(x)u}{1+b(x)u^2} \text{ in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

where $a, b \in C([0,1])$ and $b > 0$. Assume in addition that $a(x) = 0$ whenever x is near 0 or near 1. Show that bifurcation from ∞ occurs at $\lambda_0 = \pi^2$.

Solution to problem 26:
We consider

$$X \coloneqq \{u \in C^2([0,1]) \colon u(0) = u(1) = 0\}, \quad Z \coloneqq C([0,1])$$

as well as

$$F \colon \mathbb{R} \times X \to Z, F(\lambda, u) = u'' + \lambda u + \frac{a(x)u}{1+b(x)u^2}$$

which we write as $F(\lambda, u) = L(\lambda)u + R(\lambda, u)$ with

$$L(\lambda)u = u'' + \lambda u, \quad R(\lambda, u) = \frac{a(x)u}{1+b(x)u^2}$$

Claim: $\ker(L(\lambda_0)) = \mathbb{R}\phi$, $\operatorname{ran}(L(\lambda_0)) = \left\{ f \in Z \colon \int_0^1 f\phi \, \mathrm{d}x = 0 \right\}$ where $\phi(x) = \sin(\pi x)$. In particular, $L(\lambda_0)$ is a $(1,1)$-Fredholm operator and assumption (S) is satisfied.

Proof: We first calculate the kernel:

$$\begin{aligned} u \in \ker(L(\lambda_0)) &\iff u(0) = u(1) = 0, u \in C^2([0,1]), u'' + \pi^2 u = 0 \\ &\iff u(0) = u(1) = 0, u = \alpha \sin(\pi x) + \beta \cos(\pi x) \text{ for some } \alpha, \beta \in \mathbb{R} \\ &\iff u \in \mathbb{R}\phi. \end{aligned}$$

To calculate the range, we proceed as in Poblem 23. Let u_f denote the unique solution to the initial value problem

$$\begin{cases} u'' + \pi^2 u = f \text{ in } (0,1), \\ u(0) = u'(0) = 0, \end{cases}$$

where $f \in Z$. Then we have

$$\begin{aligned} \int_0^1 f\phi \, \mathrm{d}x &= \int_0^1 (u_f'' + \pi^2 u_f)\phi \, \mathrm{d}x \\ &= \left[u_f'\phi\right]_0^1 + \int_0^1 -u_f'\phi' + \pi^2 u_f \phi \, \mathrm{d}x \\ &= \left[u_f'\phi - u_f\phi'\right]_0^1 + \int_0^1 u_f\phi'' + \pi^2 u_f \phi \, \mathrm{d}x \\ &= -\phi'(1)u_f(1). \end{aligned}$$

where $\phi'(1) = -\pi \neq 0$. Hence

$$\begin{aligned} f \in \operatorname{ran}(L(\lambda_0)) &\iff \exists \alpha, \beta \in \mathbb{R} \colon \alpha \sin(\pi x) + \beta \cos(\pi x) + u_f \in Z \\ &\iff u_f(1) = 0 \\ &\iff \int_0^1 f\phi \, \mathrm{d}x = 0. \end{aligned}$$ □

Claim: $L'(\lambda_0)[\phi] \notin \operatorname{ran}(L(\lambda_0))$.

This follows from and the previous claim.

Claim: Bifurcation from infinity occurs at $\lambda_0 = \pi^2$.

We intend to apply Theorem 5.2. Assumption (S) and (T) follow from the previous claim using $L'(\lambda_0)[\phi] = \phi$. We next check assumption (R). To do so, consider

$$s\frac{a(x)\frac{u}{s}}{1+b(x)\frac{u^2}{s^2}} = \frac{s^2 a(x)u}{s^2+b(x)u^2}.$$

We need to verify that this formula admits an extension $\mathcal{R} \in C^1(\mathcal{U}; Z)$ where $\mathcal{U}$ is a suitable neighbourhood of $(\pi^2, \phi, 0)$. Let $\varepsilon > 0$ be such that $a = 0$ on both $[0, \varepsilon]$ and $[1-\varepsilon, 1]$, and let $r := \min_{[\varepsilon, 1-\varepsilon]} \phi$. We now define

$$\mathcal{U} := \mathbb{R} \times \{u \in X \colon \|u - \phi\|_\infty < r\} \times \mathbb{R},$$

$$\mathcal{R}\colon \mathcal{U} \to Z, \quad \mathcal{R}(\lambda, u, s)(x) = \begin{cases} 0, & x \in [0,\varepsilon) \cup (1-\varepsilon, 1], \\ \frac{s^2 a(x)u(x)}{s^2+b(x)u(x)^2}, & x \in [\varepsilon, 1-\varepsilon] \end{cases}$$

where we note that $u(x) = \phi(x) + u(x) - \phi(x) > r - \|u-\phi\|_\infty > 0$ for $x \in [\varepsilon, 1-\varepsilon]$, hence the fraction is well-defined and furthermore $\mathcal{R}(\lambda, u, s)$ is continuous. One can show (!) that $\mathcal{R}$ is smooth and the relevant first-order derivatives are given by $\mathcal{R}_\lambda(\lambda, u, s) = 0$ and

$$\mathcal{R}_u(\lambda, u, s)[h](x) = \begin{cases} 0, & x \in [0,\varepsilon) \cup (1-\varepsilon, 1] \\ s^2 a(x)h(x)\frac{s^2 - b(x)u(x)^2}{(s^2+b(x)u(x)^2)^2}, & x \in [\varepsilon, 1-\varepsilon]. \end{cases}$$

In particular, $\mathcal{R}(\lambda_0, \phi, 0) = 0$, $\mathcal{R}_\lambda(\lambda_0, \phi, 0) = 0$, $\mathcal{R}_u(\lambda_0, \phi, 0) = 0$.

Since all assumptions of Theorem 5.2 have been verified, we can employ the Theorem and find that Bifurcation at ∞ occurs at $\lambda = \pi^2$.

Remark: The assumption on a can be omitted. In fact, since $u \in C^1[0,1]$, $u(0) = u(1) = 0$ we have $|u(x)| \le \|u'\|_\infty \operatorname{dist}(x, \{0,1\})$. Also $\phi(x) \ge 2\operatorname{dist}(x, \{0,1\})$, so considering (λ, u, s) with $\|u - \phi\|_{C^1([0,1])} < 2$, one can again show that $\mathcal{R}$ is continuously differentiable and that the derivative is given by

$$\mathcal{R}_u(\lambda, u, s)[h](x) = s^2 a(x)h(x)\frac{s^2 - b(x)u(x)^2}{(s^2+b(x)u(x)^2)^2}, \quad x \in (0,1).$$