Karlsruhe Institute of Technology

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# Problem Sheet 11

## Bifurcation Theory

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For $x, y \in \mathbb{C}^n$ we write $x \cdot y = \sum_{k=1}^n x_k y_k$ and $\langle x, y\rangle = \overline{x} \cdot y = \sum_{k=1}^n \overline{x_k} y_k$.

We recall assumptions (H1)–(H4):

(H1) "Regularity": $f \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ with $f(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

(H2) "Simplicity": There are $\phi, \psi \in \mathbb{C}^n$ with $\langle \psi, \phi\rangle = 1$ and $\beta > 0$ such that

$$\ker(f_x(\lambda_0, 0) - \mathrm{i}\beta) = \mathrm{span}\{\phi\}, \quad \ker(f_x(\lambda_0, 0)^\top + \mathrm{i}\beta) = \mathrm{span}\{\psi\}.$$

(H3) "Nonresonance": For all $k \in \mathbb{Z}$ such that $|k| \neq 1$ we have $\ker(f_x(\lambda_0, 0) - \mathrm{i}k\beta) = \{0\}$.

(H4) "Transversality": $\mathrm{Re}(\langle f_{x\lambda}(\lambda_0, 0)\phi, \psi\rangle) \neq 0$.

**Problem 27 (Hopf bifurcation is incompatible with energy method):**
Consider the pendulum equation

$$\theta''(t) + \lambda \sin(\theta(t)) = 0 \tag{1}$$

as discussed in Chapter 2. Write (1) as a first order problem

$$x'(t) = f(\lambda, x(t)) \tag{2}$$

for $x = (\theta, \theta')$ and show that assumptions (H1)–(H4) are not satisfied for any $\lambda_0 \in \mathbb{R}$.

**Problem 28 (On simplicity(H2)):**
Let $\phi, \psi$ be as in (H2). Show that $\psi \cdot \phi = 0$.

**Problem 29:**
Prove Proposition 7.1:

Let $\xi \in \mathbb{C}^n$, $k \in \mathbb{Z}$, $t \in \mathbb{R}$. Further let $\phi, \psi$ be as in (H2) and write $A := f_x(\lambda_0, 0)$. Then

$$y_k(t) := -\tfrac{1}{\beta} \int_0^t \exp\left(\frac{t-\tau}{\beta} A\right) \xi \, \mathrm{e}^{\mathrm{i}k\tau} \, \mathrm{d}\tau$$

has the following closed form representation:

(i) $y_k(t) = \left(\mathrm{e}^{\mathrm{i}kt} I - \mathrm{e}^{\frac{t}{\beta}A}\right)(A - \mathrm{i}k\beta)^{-1}\xi$ if $|k| \neq 1$,

(ii) $y_1(t) = \left(\mathrm{e}^{\mathrm{i}t} I - \mathrm{e}^{\frac{t}{\beta}A}\right) w - \frac{t}{\beta}\mathrm{e}^{\mathrm{i}t}\langle \psi, \xi\rangle \phi$
where $w \in \mathbb{C}^n$ satisfies $(A - \mathrm{i}\beta)w = \xi - \langle \psi, \xi\rangle\phi$.