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# Solution to Problem Sheet 11

## Bifurcation Theory

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For $x, y \in \mathbb{C}^n$ we write $x \cdot y = \sum_{k=1}^n x_k y_k$ and $\langle x, y\rangle = \overline{x} \cdot y = \sum_{k=1}^n \overline{x_k} y_k$.

We recall assumptions (H1)–(H4):

(H1) "Regularity": $f \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ with $f(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

(H2) "Simplicity": There are $\phi, \psi \in \mathbb{C}^n$ with $\langle \psi, \phi\rangle = 1$ and $\beta > 0$ such that

$$\ker(f_x(\lambda_0, 0) - \mathrm{i}\beta) = \operatorname{span}\{\phi\}, \quad \ker(f_x(\lambda_0, 0)^\top + \mathrm{i}\beta) = \operatorname{span}\{\psi\}.$$

(H3) "Nonresonance": For all $k \in \mathbb{Z}$ such that $|k| \neq 1$ we have $\ker(f_x(\lambda_0, 0) - \mathrm{i}k\beta) = \{0\}$.

(H4) "Transversality": $\operatorname{Re}(\langle f_{x\lambda}(\lambda_0, 0)\phi, \psi\rangle) \neq 0$.

**Problem 27 (Hopf bifurcation is incompatible with energy method):**
Consider the pendulum equation

$$\theta''(t) + \lambda \sin(\theta(t)) = 0 \tag{1}$$

as discussed in Chapter 2. Write (1) as a first order problem

$$x'(t) = f(\lambda, x(t)) \tag{2}$$

for $x = (\theta, \theta')$ and show that assumptions (H1)–(H4) are not satisfied for any $\lambda_0 \in \mathbb{R}$.

**Solution to problem 27:**
Writing $x = (u, v)$ we can rephrase (1) as the first order problem (2) with $f(\lambda, (u, v)) := (v, -\lambda \sin(u))$. Then $f \in C^\infty(\mathbb{R} \times \mathbb{R}^2; \mathbb{R}^2)$ (in particular (H1) is satisfied) and

$$f_x(\lambda, (u, v)) = \begin{pmatrix} 0 & 1 \\ -\lambda \cos(u) & 0 \end{pmatrix}, \qquad f_{x\lambda}(\lambda, (u, v)) = \begin{pmatrix} 0 & 0 \\ -\cos(u) & 0 \end{pmatrix}.$$

So we have

$$A(\lambda) := f_x(\lambda, 0) = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}, \qquad f_{x\lambda}(\lambda, 0) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

*Case 1:* If $\lambda_0 \leq 0$, then $A(\lambda_0)$ has eigenvalues $\pm\sqrt{-\lambda_0}$, which are both real. Thus assumption (H2) is not satisfied.

*Case 2:* If $\lambda_0 > 0$, then $A(\lambda_0)$ has simple eigenvalues $\pm \mathrm{i}\sqrt{\lambda_0}$. Let $\beta := \sqrt{\lambda_0}$. We further calculate

$$\ker(A(\lambda_0) - \mathrm{i}\beta) = \operatorname{span}\{\begin{pmatrix} 1 \\ \mathrm{i}\beta \end{pmatrix}\}, \quad \ker(A(\lambda_0)^\top + \mathrm{i}\beta) = \operatorname{span}\{\begin{pmatrix} -\mathrm{i}\beta \\ 1 \end{pmatrix}\}.$$

We conclude that the nonresonance condition (H3) holds. The simplicity condition also holds, since by choosing

$$\phi = \begin{pmatrix} 1 \\ \mathrm{i}\beta \end{pmatrix} \quad \text{and} \quad \psi = \frac{1}{2}\begin{pmatrix} 1 \\ \frac{\mathrm{i}}{\beta} \end{pmatrix}$$

we have $\ker(A(\lambda_0) - \mathrm{i}\beta) = \operatorname{span}\{\phi\}$, $\ker(A(\lambda_0)^\top + \mathrm{i}\beta) = \operatorname{span}\{\psi\}$ and also $\langle\psi,\phi\rangle = 1$. So also (H2) holds.

We now verify that the Transversality condition (H4) fails:

$$\operatorname{Re}(\langle F_{x\lambda}(\lambda,0)\phi,\psi\rangle) = \tfrac{1}{2}\operatorname{Re}\left(\left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \mathrm{i}\beta \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{\mathrm{i}}{\beta} \end{pmatrix} \right\rangle\right) = \tfrac{1}{2}\operatorname{Re}\left(\tfrac{\mathrm{i}}{\beta}\right) = 0.$$

**Problem 28 (On simplicity(H2)):**
Let $\phi, \psi$ be as in (H2). Show that $\psi \cdot \phi = 0$.

**Solution to problem 28:**
Let $A \coloneqq f_x(\lambda_0, 0)$. We then calculate

$$\begin{aligned} 0 &= (A^\top + \mathrm{i}\beta)\psi \cdot \phi \\ &= \left\langle \overline{(A^\top + \mathrm{i}\beta)\psi}, \phi \right\rangle \\ &= \left\langle (A^\top - \mathrm{i}\beta)\overline{\psi}, \phi \right\rangle \\ &= \left\langle \overline{\psi}, (A + \mathrm{i}\beta)\phi \right\rangle \\ &= \left\langle \overline{\psi}, 2\mathrm{i}\beta\phi \right\rangle \\ &= 2\mathrm{i}\beta\psi \cdot \phi. \end{aligned}$$

As $\beta > 0$, we conclude $\psi \cdot \phi = 0$.

**Problem 29:**
Prove Proposition 7.1:

> Let $\xi \in \mathbb{C}^n$, $k \in \mathbb{Z}$, $t \in \mathbb{R}$. Further let $\phi, \psi$ be as in (H2) and write $A \coloneqq f_x(\lambda_0, 0)$. Then
>
> $$y_k(t) \coloneqq -\tfrac{1}{\beta}\int_0^t \exp\left(\frac{t-\tau}{\beta}A\right)\xi\, \mathrm{e}^{\mathrm{i}k\tau}\,\mathrm{d}\tau$$
>
> has the following closed form representation:
>
> (i) $y_k(t) = \left(\mathrm{e}^{\mathrm{i}kt} I - \mathrm{e}^{\frac{t}{\beta}A}\right)(A - \mathrm{i}k\beta)^{-1}\xi$ if $|k| \neq 1$,
>
> (ii) $y_1(t) = \left(\mathrm{e}^{\mathrm{i}t} I - \mathrm{e}^{\frac{t}{\beta}A}\right)w - \frac{t}{\beta}\mathrm{e}^{\mathrm{i}t}\langle\psi,\xi\rangle\phi$
> where $w \in \mathbb{C}^n$ satisfies $(A - \mathrm{i}\beta)w = \xi - \langle\psi,\xi\rangle\phi$.

**Solution to problem 29:**

(i) For $|k| \neq 1$ let $\eta \coloneqq (A - \mathrm{i}k\beta)^{-1}\xi$. Then we have

$$\begin{aligned} y_k(t) &= e^{\mathrm{i}kt}\int_0^t -\frac{1}{\beta}\exp\left(\frac{t-\tau}{\beta}A + \mathrm{i}k(\tau - t)\right)\xi\,\mathrm{d}\tau \\ &= e^{\mathrm{i}kt}\int_0^t -\frac{1}{\beta}\exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}k\beta)\right)\xi\,\mathrm{d}\tau \\ &= e^{\mathrm{i}kt}\int_0^t -\frac{1}{\beta}\exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}k\beta)\right)(A - \mathrm{i}k\beta)\eta\,\mathrm{d}\tau \\ &= e^{\mathrm{i}kt}\left[\exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}k\beta)\right)\eta\right]_0^t \\ &= e^{\mathrm{i}kt}\left(\eta - \exp\left(\frac{t}{\beta}(A - \mathrm{i}k\beta)\right)\eta\right) \\ &= e^{\mathrm{i}kt}\eta - \exp\left(\frac{t}{\beta}A\right)\eta \\ &= \left(e^{\mathrm{i}kt}I - e^{\frac{t}{\beta}A}\right)(A - \mathrm{i}k\beta)^{-1}\xi. \end{aligned}$$

(ii) For $k = 1$ we similarly get

$$\begin{aligned} y_1(t) &= e^{\mathrm{i}t} \int_0^t -\frac{1}{\beta} \exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}\beta)\right)((A-\mathrm{i}\beta)w + \langle\psi,\xi\rangle\phi)\,\mathrm{d}\tau \\ &= e^{\mathrm{i}t} \int_0^t -\frac{1}{\beta} \exp\left(\frac{t-\tau}{\beta}(A - \mathrm{i}\beta)\right)(A-\mathrm{i}\beta)w - \frac{1}{\beta}\langle\psi,\xi\rangle\phi\,\mathrm{d}\tau \\ &= e^{\mathrm{i}t}\left[\exp\left(\frac{t-\tau}{\beta}(A-\mathrm{i}\beta)\right)w - \frac{\tau}{\beta}\langle\psi,\xi\rangle\phi\right]_0^t \\ &= e^{\mathrm{i}t}w - \exp\left(\frac{t}{\beta}A\right)w - \frac{t}{\beta}e^{\mathrm{i}t}\langle\psi,\xi\rangle\phi \\ &= \left(e^{\mathrm{i}t}I - e^{\frac{t}{\beta}A}\right)w - \frac{t}{\beta}e^{\mathrm{i}t}\langle\psi,\xi\rangle\phi. \end{aligned}$$

Here we used that $\phi \in \ker(A - \mathrm{i}\beta)$ and hence

$$\exp\left(\frac{t-\tau}{\beta}(A-\mathrm{i}\beta)\right)\phi = \sum_{k=0}^{\infty}\frac{(t-\tau)^k}{\beta^k k!}(A-\mathrm{i}\beta)^k\phi = \phi + \underbrace{\sum_{k=1}^{\infty}\frac{(t-\tau)^k}{\beta^k k!}(A-\mathrm{i}\beta)^k\phi}_{=0} = \phi.$$