Karlsruhe Institute of Technology

Institute for Analysis
PD Dr. Rainer Mandel
M.Sc. Sebastian Ohrem

Solution to Problem Sheet 12

Bifurcation Theory

Winter Semester 2022/23

6.2.2023

Problem 30:
A gradient system is a dynamical system of the form

$$x' = -\nabla_x V(\lambda, x)$$

for a given function $V \in C^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$.

(a) Prove that gradient systems cannot have non-constant periodic solutions.

(b) Let $V \in C^2(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$. Show that Hopf bifurcation cannot occur in gradient systems.

(c) Consider the dynamical system

$$\begin{cases} x_1' = 2 - x_1 - \lambda x_2^2, \\ x_2' = 5 - x_2 - 2\lambda x_1 x_2. \end{cases} \tag{1}$$

Are there non-constant periodic solutions to (1)?

Solution to problem 30:

(a) We suppose that there is a non-constant periodic solution x with period $T > 0$. Then we have

$$\begin{aligned} 0 &= V(\lambda, x(T)) - V(\lambda, x(0)) \\ &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} V(\lambda, x(t))\,\mathrm{d}t \\ &= \int_0^T \nabla_x V(\lambda, x(t)) \cdot x'(t)\,\mathrm{d}t \\ &= \int_0^T -\|x'(t)\|^2\,\mathrm{d}t < 0, \end{aligned}$$

since $x' \neq 0$, a contradiction.

(b) Define $f(\lambda, x) = -\nabla_x V(\lambda, x)$. Then we have

$$\frac{\partial}{\partial x_i} f(\lambda, x) = \frac{\partial}{\partial x_i}(-\nabla_x V(\lambda, x)).$$

Hence, we observe

$$f_x(\lambda, x) = -H_V(\lambda, x),$$

where H_V denotes the Hessian of V. Thus,

$$f_x(\lambda_0, 0) = -H_V(\lambda_0, 0)$$

is symmetric and has only real-valued eigenvalues. Hence the simplicity condition (H2) can not be fulfilled and Hopf bifurcation is not possible.

(c) Define $V : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, $V(\lambda, x) := -2x_1 - 5x_2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \lambda x_1 x_2^2$. Then we calculate

$$\nabla_x V(\lambda, x) = \begin{pmatrix} -2 + x_1 + \lambda x_2^2 \\ -5 + x_2 + 2\lambda x_1 x_2. \end{pmatrix}$$

Hence (1) is of the form $x' = -\nabla_x V(\lambda, x)$ which means by part (a) that no nontrivial periodic solutions exist.

Problem 31:
For $g \in C^2(\mathbb{R}^3; \mathbb{R})$ consider the differential equation

$$-u'' = g(\lambda, u, u'). \tag{2}$$

Write (2) as a two-dimensional system

$$y' = f(\lambda, y) \tag{3}$$

in the variable $y = (u, u')$. Under which conditions on g are the assumptions (H1)–(H4) satisfied for some given $\lambda_0 \in \mathbb{R}$, $\beta > 0$?

Solution to problem 31:
We rewrite our differential equation as a two-dimensional system in the variable $y = (u, u')$, i.e.

$$y' = f(\lambda, y)$$

where

$$f(\lambda, y) = \begin{pmatrix} y_2 \\ -g(\lambda, y_1, y_2) \end{pmatrix}.$$

Claim: (H1) holds if and only if $g \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g(\lambda, 0, 0) = 0$ for all $\lambda \in \mathbb{R}$.

Proof: Then we observe that $f \in C^2(\mathbb{R}^2, \mathbb{R})$ holds if and only if $g \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $f(\lambda, 0) = 0$ holds if and only if $g(\lambda, 0, 0) = 0$. □

In the following, we use the symbols $u = y_1, p = y_2$. So g_u denote the derivative of g w.r.t the variable y_1, g_p the derivative w.r.t. y_2. Also we assume that (H1) holds.

Claim: Assumption (H2) holds if and only if $g_u(\lambda_0, 0, 0) = \beta^2$, $g_p(\lambda_0, 0, 0) = 0$. In this case, ϕ, ψ can be chosen as $\phi = \begin{pmatrix} 1 \\ \mathrm{i}\beta \end{pmatrix}$ and $\psi = \frac{1}{2\beta}\begin{pmatrix} \beta \\ \mathrm{i} \end{pmatrix}$.

Proof: We calculate

$$f_x(\lambda_0, 0) = \begin{pmatrix} 0 & 1 \\ -g_u(\lambda_0, 0, 0) & -g_p(\lambda_0, 0, 0) \end{pmatrix}$$

and hence

$$f_x(0, \lambda_0) - \mathrm{i}\beta = \begin{pmatrix} -\mathrm{i}\beta & 1 \\ -g_u(\lambda_0, 0, 0) & -g_p(\lambda_0, 0, 0) - \mathrm{i}\beta \end{pmatrix}$$

and

$$f_x(0, \lambda_0)^T + \mathrm{i}\beta = \begin{pmatrix} \mathrm{i}\beta & -g_u(\lambda_0, 0, 0) \\ 1 & -g_p(\lambda_0, 0, 0) + \mathrm{i}\beta \end{pmatrix}.$$

Firstly $\det(f_u(\lambda_0, 0) - \mathrm{i}\beta) = -\beta^2 + \mathrm{i}\beta g_p(\lambda_0, 0, 0) + g_u(\lambda_0, 0, 0)$ is zero if and only if $g_p(\lambda_0, 0, 0) = 0$ and $g_u(\lambda_0, 0, 0) = \beta^2$.

In this case, the relevant kernels are given by

$$\ker(f_u(\lambda_0,0)-\mathrm{i}\beta) = \ker\begin{pmatrix}-\mathrm{i}\beta & 1\\ \beta^2 & -\mathrm{i}\beta\end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix}1\\ \mathrm{i}\beta\end{pmatrix}\right\},$$
$$\ker\big(f_u(\lambda_0,0)^\top+\mathrm{i}\beta\big) = \ker\begin{pmatrix}\mathrm{i}\beta & \beta^2\\ 1 & \mathrm{i}\beta\end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix}\beta\\ \mathrm{i}\end{pmatrix}\right\}.$$

With $\phi := \begin{pmatrix}1\\ \mathrm{i}\beta\end{pmatrix}$ and $\psi := \frac{1}{2\beta}\begin{pmatrix}\beta\\ \mathrm{i}\end{pmatrix}$ we also have

$$\langle \phi,\psi\rangle = \overline{\phi}\cdot\psi = \begin{pmatrix}1\\ -\mathrm{i}\beta\end{pmatrix}\cdot\frac{1}{2\beta}\begin{pmatrix}\beta\\ \mathrm{i}\end{pmatrix} = 1.$$ □

In the following, assume that (H2) holds.

Claim: Assumption (H3) is fulfilled.

Proof: $f_x(\lambda_0,0,0)$ is a 2×2 matrix. As it has the eigenvalues $\mathrm{i}\beta$ and $-\mathrm{i}\beta$, it can have no further eigenvalues. □

In the following, assume that (H3) holds.

Claim: Assumption (H4) is fulfilled if and only if $g_{p\lambda}(\lambda_0,0,0)\neq 0$.

Proof: We observe

$$f_{x\lambda}(0,0,\lambda_0) = \begin{pmatrix}0 & 0\\ -g_{u\lambda}(0,0,\lambda_0) & -g_{p\lambda}(0,0,\lambda_0)\end{pmatrix}$$

and

$$\begin{aligned}\langle f_{x\lambda}(0,0,\lambda_0)\phi,\psi\rangle &= \overline{f_{x\lambda}(0,0,\lambda_0)\phi}\cdot\psi\\ &= \begin{pmatrix}0\\ -g_{x\lambda}(0,0,\lambda_0)+\mathrm{i}\beta g_{p\lambda}(0,0,\lambda_0)\end{pmatrix}\cdot\frac{1}{2\beta}\begin{pmatrix}\beta\\ \mathrm{i}\end{pmatrix}\\ &= \frac{-\mathrm{i}}{2\beta}g_{x\lambda}(0,0,\lambda_0)-\frac{1}{2}g_{p\lambda}(0,0,\lambda_0).\end{aligned}$$

Finally,

$$\operatorname{Re}(\langle f_{x\lambda}(0,0,\lambda_0)\phi,\psi\rangle_{\mathbb{C}^n}) = -\frac{1}{2}g_{p\lambda}(0,0,\lambda_0)\neq 0$$

if and only if $g_{p\lambda}(0,0,\lambda_0)\neq 0$ which closes the proof. □