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Problem Sheet 13

Bifurcation Theory

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Problem 32:
For $f \in C(\mathbb{R};\mathbb{R})$ and $k \in \mathbb{N}$ consider

$$y^{(k)}(t) = f(y(t)) \tag{1}$$

(a) Show that (1) does not admit non-constant periodic solutions when k is odd.
Hint: Integrate $f(y(t))y'(t)$.

(b) Does the previous statement hold when k is even?

Problem 33:
For fixed $\gamma \in (0, 3 - 2\sqrt{2})$ consider the predator-prey problem

$$\begin{cases} x' = x - x^2 - \frac{xy}{\gamma+\lambda x}, \\ y' = -y + \frac{xy}{\gamma+\lambda x} \end{cases} \tag{2}$$

for two species x (prey) and y (predator), where $\lambda \in (0, 1-\gamma)$.

(a) Show that (2) has an equilibrium (i.e. time-independent) solution $(x_\star(\lambda), y_\star(\lambda))$ in $(0,\infty)^2$ for each $\lambda \in (0, 1-\gamma)$.

(b) Show that nontrivial periodic solutions of (2) bifurcate from the equilibrium solutions. You may proceed as follows:
   (i) Rewrite (2) as a problem $(p', q') = f(\lambda, p, q)$ in the shifted variables $p = x - x_\star(\lambda)$, $q = y - y_\star(\lambda)$.
   (ii) With $A(\lambda) := f_{(p,q)}(\lambda, 0, 0)$, show that there exist two values $\lambda_0, \lambda_1 \in (0, 1-\gamma)$ such that $A(\lambda)$ has purely imaginary eigenvalues.
   (iii) Show that nontrivial periodic solutions of (2) bifurcate from $(\lambda_0, 0, 0)$ as well as from $(\lambda_1, 0, 0)$.

Problem 34:
Determine the Hopf bifurcation points $(\lambda_0, 0) \in \mathbb{R}^3 \times \mathbb{R}$ of the nonlinear system

$$\begin{cases} x_1' = (\lambda - 1)x_1 - x_2 + x_1x_3, \\ x_2' = x_1 + (\lambda - 1)x_2 + x_2x_3, \\ x_3' = \lambda x_3 - (x_1^2 + x_2^2 + x_3^2). \end{cases}$$