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Solution to Problem Sheet 13

Bifurcation Theory

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Problem 32:
For $f \in C(\mathbb{R};\mathbb{R})$ and $k \in \mathbb{N}$ consider

$$y^{(k)}(t) = f(y(t)) \tag{1}$$

(a) Show that (1) does not admit non-constant periodic solutions when k is odd.
Hint: Integrate $f(y(t))y'(t)$.

(b) Does the previous statement hold when k is even?

Solution to problem 32:

(a) *Proof.* Let $k = 2n+1$ be odd. Assume for a contradiction that (1) has a non-constant solution y with period $T > 0$. Let F denote a primitive of f. We calculate

$$0 = [F(y(s))]_{s=0}^{T} = \int_0^T f(y(s))y'(s)\,\mathrm{d}s = \int_0^T y^{(2n+1)}y'(s)\,\mathrm{d}s$$

Using integration by parts n times, where boundary terms cancel by the peridicity of y, we find

$$0 = \int_0^T y^{(2n+1)}y'(s)\,\mathrm{d}s = (-1)^n \int_0^T y^{(n+1)}(s)y^{(n+1)}(s)\,\mathrm{d}s.$$

Hence $y^{(n+1)} = 0$, so y is a polynomial. As y is periodic, y must be constant, a contradiction. ☐

(b) *Claim:* For even $k = 2n$, (1) can have non-constant solutions.

Proof. For example, $y(t) = \sin(t)$ solves (1) with right-hand side $f(y) = (-1)^n y$. ☐

Problem 33:
For fixed $\gamma \in (0, 3 - 2\sqrt{2})$ consider the predator-prey problem

$$\begin{cases} x' = x - x^2 - \frac{xy}{\gamma+\lambda x}, \\ y' = -y + \frac{xy}{\gamma+\lambda x} \end{cases} \tag{2}$$

for two species x (prey) and y (predator), where $\lambda \in (0, 1-\gamma)$.

(a) Show that (2) has an equilibrium (i.e. time-independent) solution $(x_\star(\lambda), y_\star(\lambda))$ in $(0,\infty)^2$ for each $\lambda \in (0, 1-\gamma)$.

(b) Show that nontrivial periodic solutions of (2) bifurcate from the equilibrium solutions. You may proceed as follows:
   (i) Rewrite (2) as a problem $(p', q') = f(\lambda, p, q)$ in the shifted variables $p = x - x_\star(\lambda)$, $q = y - y_\star(\lambda)$.
   (ii) With $A(\lambda) := f_{(p,q)}(\lambda, 0, 0)$, show that there exist two values $\lambda_0, \lambda_1 \in (0, 1-\gamma)$ such that $A(\lambda)$ has purely imaginary eigenvalues.
   (iii) Show that nontrivial periodic solutions of (2) bifurcate from $(\lambda_0, 0, 0)$ as well as from $(\lambda_1, 0, 0)$.

Solution to problem 33:

(a) For $x, y > 0$ we calculate

$$0 = -y + \frac{xy}{\gamma + \lambda x} \iff 1 = \frac{x}{\gamma + \lambda x} \iff x = \frac{\gamma}{1-\lambda},$$
$$0 = x - x^2 - \frac{xy}{\gamma + \lambda x} \iff y = (1-x)(\gamma + \lambda x),$$

so with

$$x_\star(\lambda) := \frac{\gamma}{1-\lambda} \quad \text{and} \quad y_\star(\lambda) := (1 - x_\star(\lambda))(\gamma + \lambda x_\star(\lambda)) = \frac{\gamma(1-\gamma-\lambda)}{(1-\lambda)^2}$$

we have found our equilibrium point.

(b) We proceed as suggested in the problem.

(i) Corresponding to (2) we define

$$g(\lambda, x, y) = \begin{pmatrix} x - x^2 - \frac{xy}{\gamma+\lambda x} \\ -y + \frac{xy}{\gamma+\lambda x} \end{pmatrix}.$$

and thus $f(\lambda, p, q) := g(\lambda, x_\star(\lambda) + p, y_\star(\lambda) + q)$. In the following, we do not need a closed form representation of f.

(ii) As a next step, we calculate the linearization $A(\lambda)$. Note that

$$g_{(x,y)}(\lambda, x, y) = \begin{pmatrix} 1 - 2x - \frac{\gamma y}{(\gamma+\lambda x)^2} & -\frac{x}{\gamma+\lambda x} \\ \frac{\gamma y}{(\gamma+\lambda x)^2} & -1 + \frac{x}{\gamma+\lambda x} \end{pmatrix},$$

At $x = x_\star(\lambda)$ and $y = y_\star(\lambda)$ we can further simplify:

$$\frac{x}{\gamma + \lambda x} = 1,$$
$$\frac{\gamma y}{(\gamma+\lambda x)^2} = \frac{\gamma(1-x)(\gamma+\lambda x)}{(\gamma+\lambda x)^2} = \frac{\gamma(1-x)}{x} = (1-\lambda)(1-x) = 1 - \gamma - \lambda,$$
$$1 - 2x - \frac{\gamma y}{(\gamma+\lambda x)^2} = \lambda + \gamma - 2x = -\frac{\lambda^2 - (1-\gamma)\lambda + \gamma}{1-\lambda}.$$

Using this we obtain

$$A(\lambda) = f_{(p,q)}(\lambda, 0, 0) = g_{(x,y)}(\lambda, x_\star(\lambda), y_\star(\lambda)) = \begin{pmatrix} -\frac{\lambda^2 + (\gamma-1)\lambda + \gamma}{1-\lambda} & -1 \\ 1 - \gamma - \lambda & 0 \end{pmatrix}.$$

This Matrix satisfies

$$\det(A(\lambda)) = 1 - \gamma - \lambda > 0, \qquad \operatorname{tr}(A(\lambda)) = -\frac{\lambda^2 + (\gamma-1)\lambda + \gamma}{1-\lambda} =: -\frac{p(\lambda)}{1-\lambda}$$

Further, the map p has the two zeros

$$\lambda_0 = \frac{1 - \gamma - \sqrt{1 - 6\gamma + \gamma^2}}{2}, \qquad \lambda_1 = \frac{1 - \gamma + \sqrt{1 - 6\gamma + \gamma^2}}{2},$$

which ar real and distinct since $1 - 6\gamma + \gamma^2 > 0$. This now holds by our assumption $\gamma < 3 - 2\sqrt{2}$. Moreover, we have $\sqrt{1 - 6\gamma + \gamma^2} < 1 - \gamma$, showing that λ_0, λ_1 both lie in $(0, 1-\gamma)$.

(iii) It remains to verify the assumptions (H1)-(H4) of Hopf bifurcation. Before we do this, we note that f is not defined on $\mathbb{R}^3$ but only on

$$\Omega = \{(\lambda, x, y) \in \mathbb{R}^3 \colon 0 < \lambda < 1 - \gamma, \gamma + \lambda x > 0\}.$$

Nevertheless, the Hopf bifurcation theorem can be applied without changes. We only provide the argument for λ_0.

(H1) is clear since f is a rational function and thus C^∞ on Ω, and $f(\lambda, 0, 0) = 0$ holds by construction.

Recall that a 2×2 matrix A has eigenvalues

$$\frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}}{2} \tag{3}$$

It follows that $A(\lambda_0)$ has the two distict eigenvalues $\pm \mathrm{i}\beta$ where $\beta = \sqrt{\det(A(\lambda_0))} = \sqrt{1-\gamma-\lambda_0}$. From the fact that $A(\lambda_0)$ is a 2×2 matrix, two simple observations follow: First, $\pm\mathrm{i}\beta$ must be algebraically simple eigenvalues (i.e. (H2) holds) and there exist no further eigenvalues (i.e. (H3) holds).

To show that (H4) holds, recall that in the exercise session we proved (H4) is equivalent to

$$\operatorname{Re}\mu'(\lambda_0) \neq 0$$

where $\mu(\lambda)$ is an eigenvalue of $A(\lambda)$ such that $\mu(\lambda_0) = \mathrm{i}\beta$ and $\lambda \mapsto \mu(\lambda)$ is differentiable.

For λ close to λ_0, from (3) we obtain $\operatorname{Re}\mu(\lambda) = \frac{1}{2}\operatorname{tr}(A(\lambda))$. Using

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\operatorname{tr}(A(\lambda)) = -\frac{(1-\lambda)p'(\lambda) + p(\lambda)}{(1-\lambda)^2},$$

and $p(\lambda) = (\lambda-\lambda_0)(\lambda-\lambda_1)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\operatorname{tr}(A(\lambda_0)) = -\frac{(1-\lambda_0)(\lambda_0-\lambda_1)+0}{(1-\lambda_0)^2} \neq 0.$$

So also (H4) holds. Now the Hopf bifurcation theorem yields existence of nontrivial solutions bifurcating from $(\lambda_0, 0, 0)$.

Problem 34:

Determine the Hopf bifurcation points $(\lambda_0, 0) \in \mathbb{R}^3 \times \mathbb{R}$ of the nonlinear system

$$\begin{cases} x_1' = (\lambda-1)x_1 - x_2 + x_1x_3, \\ x_2' = x_1 + (\lambda-1)x_2 + x_2x_3, \\ x_3' = \lambda x_3 - (x_1^2 + x_2^2 + x_3^2). \end{cases}$$

Solution to problem 34:

Assertion: $(0, \lambda_0)$ *is a Hopf bifurcation point if and only if* $\lambda_0 = 1$.

We define

$$f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3, \quad f(x,\lambda) := \begin{pmatrix} (\lambda-1)x_1 - x_2 + x_1x_3 \\ x_1 + (\lambda-1)x_2 + x_2x_3 \\ \lambda x_3 - (x_1^2+x_2^2+x_3^2) \end{pmatrix}$$

Then $f(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$ and $f \in C^2(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ with

$$f_x(x,\lambda) = \begin{pmatrix} \lambda - 1 + x_3 & -1 & x_1 \\ 1 & \lambda - 1 + x_3 & x_2 \\ -2x_1 & -2x_2 & \lambda - 2x_3 \end{pmatrix}, \quad f_{x\lambda}(x,\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = id$$

for $x_1, x_2, x_3, \lambda \in \mathbb{R}$, and in particular, for $\lambda_0 \in \mathbb{R}$,

$$f_x(0,\lambda_0) = \begin{pmatrix} \lambda_0 - 1 & -1 & 0 \\ 1 & \lambda_0 - 1 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix}, \quad f_{x\lambda}(0,\lambda_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = id$$

The eigenvalues of $f_x(0,\lambda_0)$ are given by λ_0 and $\lambda_0 - 1 \pm \mathrm{i}$ and the eigenvalues of $f_x(0,\lambda_0)^T$ are also λ_0 and $\lambda_0 - 1 \pm \mathrm{i}$. Hence, simplicity (H2) tells us that Hopf bifurcation can only occur for $\lambda_0 = 1$. (Here $\beta = 1$.) Then,

$$\ker(f_x(0,\lambda_0) - \mathrm{i}) = \operatorname{span}\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \right\}$$

and

$$\ker(f_x(0,\lambda_0)^T + \mathrm{i}) = \operatorname{span}\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \right\}.$$

With $\phi := \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$ and $\psi := \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$ we then have

$$\langle \psi, \phi \rangle_{\mathbb{C}^n} = \psi \cdot \overline{\phi} = 1$$

and hence (H2) is fulfilled. Additionally we have for all $k \in \mathbb{Z}$ with $|k| \neq 1$

$$f_x(0,\lambda_0) - \mathrm{i}k = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i}k & -1 & 0 \\ 1 & -\mathrm{i}k & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

i.e. $\det(f_x(0,\lambda_0) - \mathrm{i}k) = (-\mathrm{i}k)^2 + 1 = -k^2 + 1 \neq 0$ which shows

$$\ker(f_x(0,\lambda_0) - \mathrm{i}k) = \{0\}$$

and thus proves the nonresonance condition (H3). Finally, we oberserve

$$\mathrm{Re}(\langle f_{x\lambda}(0,\lambda_0)\phi, \psi \rangle_{\mathbb{C}^n}) = \mathrm{Re}(\langle \phi, \psi \rangle_{\mathbb{C}^n}) = 1 \neq 0$$

which is exactly the transversality condition (H4) and closes the proof.