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Problem Sheet 1

Bifurcation Theory

Winter Semester 2022/23

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Problem 1:
Draw bifurcation diagrams for the following equations:

(a) $x^3 + 2\lambda x^2 + \lambda^3 x = 0$ $(\lambda, x \in \mathbb{R})$,

(b) $x + \sinh(\lambda x) = 0$ $(\lambda, x \in \mathbb{R})$.

Problem 2:
Consider the function $u_1 : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{\sqrt{2}}{\cosh(x)}$.

(a) Prove that $u_1 \in W^{2,q}(\mathbb{R})$ for all $q \in [1, \infty]$.

(b) Prove that u_1 solves the ODE $-u'' + u - u^3 = 0$ on $\mathbb{R}$.

(c) Find a nontrivial family of solutions $\mathcal{T} := \{(u_\lambda, \lambda) : \lambda > 0\}$ of

$$\begin{cases} -u'' + \lambda u - u^3 = 0, \\ u \in W^{2,q}(\mathbb{R}) \end{cases}$$

and, for each $q \in [1, \infty]$, decide whether it bifurcates from the trivial branch at $(0, 0)$ with respect to $\|\cdot\|_{W^{2,q}(\mathbb{R})}$.

Problem 3:
Consider the boundary value problem

$$\begin{cases} u'' + \lambda(u - u^3) = 0 \quad \text{in } (0, \pi), \\ u(0) = u(\pi) = 0. \end{cases} \tag{1}$$

Using the Energy Method, prove that for $\alpha \in (0, 1)$ there exists a nontrivial positive solution $(u_\alpha, \lambda_\alpha)$ of (1) with $\|u_\alpha\|_\infty = \alpha$. Furthermore show that $(u_\alpha, \lambda_\alpha)$ bifurcates (with respect to $\|\cdot\|_\infty$ norm) from the trivial branch at $\lambda_0 = 1$.