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Solution to Problem Sheet 1

Bifurcation Theory

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Problem 1:
Draw bifurcation diagrams for the following equations:

(a) $x^3 + 2\lambda x^2 + \lambda^3 x = 0$ $(\lambda, x \in \mathbb{R})$,

(b) $x + \sinh(\lambda x) = 0$ $(\lambda, x \in \mathbb{R})$.

Solution to problem 1:
Let us first note that both equations are satisfied by the trivial solution family $\{(0, \lambda) : \lambda \in \mathbb{R}\}$. We now aim to find (nontrivial) solutions $(x, \lambda) \in \mathbb{R} \times \mathbb{R}$, $x \neq 0$.

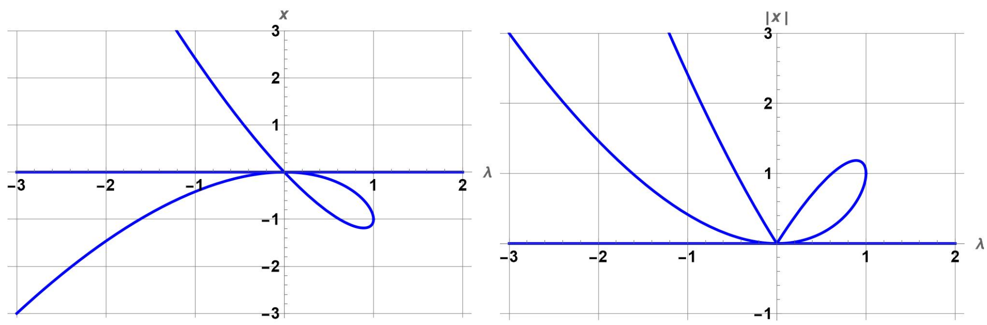
(a) We define the auxiliary function

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad f(x, \lambda) := x^2 + 2\lambda x + \lambda^3$$

and calculate its zeros explicitly:

$$\begin{aligned} 0 = f(x, \lambda) \quad &\Leftrightarrow \quad 0 = (x + \lambda)^2 + \lambda^3 - \lambda^2 \\ &\Leftrightarrow \quad \lambda \leq 1, \quad x \in \left\{-\lambda \pm \sqrt{\lambda^2 - \lambda^3}\right\}. \end{aligned}$$

This yields the following bifurcation diagram which shows that $(0, 0)$ is a bifurcation point:



(b) For every $\lambda \in \mathbb{R}$, we introduce the smooth function

$$g_\lambda : \mathbb{R} \to \mathbb{R}, \quad g_\lambda(x) := x + \sinh(\lambda x).$$

We already know that $g_\lambda(0) = 0$, which corresponds to the trivial branch. Further,

$$g'_\lambda(x) = 1 + \lambda \cosh(\lambda x) \qquad (x \in \mathbb{R})$$

implies (due to $\cosh(y) > 1, y \in \mathbb{R} \setminus \{0\}$) that g_λ is strictly monotone for $\lambda \geq 0$ and $\lambda \leq -1$, so in these cases, 0 is the only zero of g_λ. In the remaining cases, we prove:

(1) <u>*Claim*</u>*:* For $-1 < \lambda < 0$, there exists $x_\lambda > 0$ with the property that

$$g_\lambda^{-1}(\{0\}) = \{-x_\lambda, 0, x_\lambda\}.$$

Proof: Let $-1 < \lambda < 0$. As the function g_λ is odd, we only have to prove that it has exactly one positive zero x_λ.

Existence: As $\lambda < 0$, we conclude $\lim_{x\to\infty} g_\lambda(x) = -\infty$. Further, $\lambda > -1$ implies that $g'_\lambda(0) = 1 + \lambda > 0$, and hence, there exists $\delta_\lambda > 0$ with $g_\lambda(\delta_\lambda) = \max_{x>0} g_\lambda(x) > 0$. Since g_λ is continuous, the intermediate value theorem now yields the existence of a zero $x_\lambda > \delta_\lambda$.

Uniqueness: This is a consequence of the fact that g_λ is strictly concave on $(0, \infty)$ due to

$$g''_\lambda(x) = \lambda^2 \sinh(\lambda x) < 0 \qquad (x > 0).$$

(Hence, the first derivative is strictly decreasing, and g_λ has at most one critical point in $(0, \infty)$, which is $\delta_\lambda \in (0, x_\lambda)$. We infer that g_λ is strictly monotone on both $(0, \delta_\lambda)$ and (δ_λ, ∞), which proves the assertion.) □

(2) *Claim:* The mapping $\lambda \mapsto x_\lambda$ $(-1 < \lambda < 0)$ is continuous.

Proof: This is a consequence of the Implicit Function Theorem, which can be applied since

$$g_\lambda(x_\lambda) = 0, \qquad g'_\lambda(x_\lambda) < 0$$

(where the latter inequality results from strict concavity, see above). □

(3) *Claim:* We have the following one-sided limits:

$$\lim_{\lambda\to -1^+} x_\lambda = 0, \qquad \lim_{\lambda\to 0^-} x_\lambda = \infty.$$

Proof: For $\lambda \in (-1, 0)$, we estimate by power series expansion:

$$x_\lambda = -\sinh(\lambda x_\lambda) = \sinh(|\lambda| x_\lambda) \geq |\lambda| x_\lambda + \frac{1}{6}|\lambda|^3 x_\lambda^3,$$

which yields

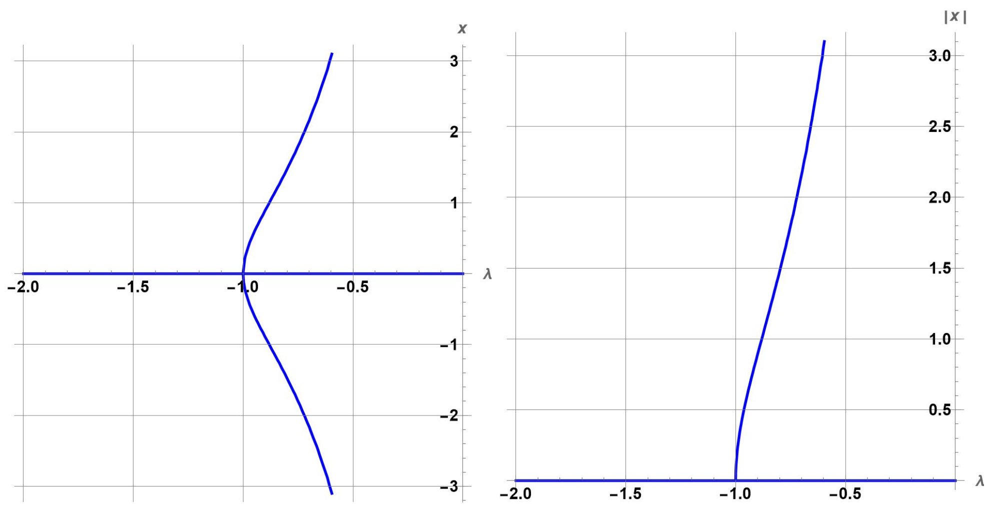
$$x_\lambda^2 \leq \frac{6}{|\lambda|^3}(1 - |\lambda|) \to 0 \qquad (\lambda \to -1,\ \lambda > -1).$$

On the other hand, we recall that $g'_\lambda(x_\lambda) < 0$ and obtain by inserting $g_\lambda(x_\lambda) = 0$

$$0 > 1 - |\lambda| \cosh(\lambda x_\lambda) = 1 - |\lambda| \sqrt{1 + \sinh^2(\lambda x_\lambda)} = 1 - |\lambda| \sqrt{1 + x_\lambda^2}.$$

This can only hold in the limit $\lambda \to 0$, $\lambda < 0$ if, at the same time, $x_\lambda \to \infty$. □

This yields the following bifurcation diagram:



Problem 2:
Consider the function $u_1 : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{\sqrt{2}}{\cosh(x)}$.

(a) Prove that $u_1 \in W^{2,q}(\mathbb{R})$ for all $q \in [1,\infty]$.

(b) Prove that u_1 solves the ODE $-u'' + u - u^3 = 0$ on $\mathbb{R}$.

(c) Find a nontrivial family of solutions $\mathcal{T} \coloneqq \{(u_\lambda, \lambda) :\ \lambda > 0\}$ of

$$\begin{cases} -u'' + \lambda u - u^3 = 0, \\ u \in W^{2,q}(\mathbb{R}) \end{cases} \tag{1}$$

and, for each $q \in [1,\infty]$, decide whether it bifurcates from the trivial branch at $(0,0)$ with respect to $\|\cdot\|_{W^{2,q}(\mathbb{R})}$.

Solution to problem 2:

Proof: (a) Clearly, u_1 is a smooth function. For $x \in \mathbb{R}$, we have (with $\sinh^2(x) + 1 = \cosh^2(x)$)

$$\begin{aligned} u_1'(x) &= -\frac{\sqrt{2}}{\cosh^2(x)} \cdot \sinh(x), \\ u_1''(x) &= \frac{2\sqrt{2}}{\cosh^3(x)} \cdot \sinh^2(x) - \frac{\sqrt{2}}{\cosh^2(x)} \cdot \cosh(x) = \frac{\sqrt{2}}{\cosh^3(x)} \cdot \left[2\sinh^2(x) - \cosh^2(x)\right] \\ (\diamondsuit) \qquad &= \frac{\sqrt{2}}{\cosh^3(x)} \cdot \left[\cosh^2(x) - 2\right] = u_1(x) - u_1(x)^3. \end{aligned}$$

As $u_1 \in C^\infty(\mathbb{R})$, weak derivatives of second (in fact, of every) order exist and agree with the classical ones computed above.

Let $q \in [1,\infty]$. It remains to check integrability, i.e. whether $u_1, u_1', u_1'' \in L^q(\mathbb{R})$. We exploit that, for $x \in \mathbb{R}$,

$$\cosh(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} \geq \frac{1}{2}\mathrm{e}^{|x|} \quad \text{and} \quad |\sinh(x)| = \frac{|\mathrm{e}^x - \mathrm{e}^{-x}|}{2} \leq \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} = \cosh(x),$$

and estimate as follows:

$$\begin{aligned} |u_1(x)| &= \frac{\sqrt{2}}{\cosh(x)} \leq 2\sqrt{2}\,\mathrm{e}^{-|x|}, \\ |u_1'(x)| &= \frac{\sqrt{2}}{\cosh^2(x)} \cdot \sinh(|x|) \leq \frac{\sqrt{2}}{\cosh(x)} \leq 2\sqrt{2}\,\mathrm{e}^{-|x|}, \\ |u_1''(x)| &= \frac{\sqrt{2}}{\cosh^3(x)} \cdot |\cosh^2(x) - 2| \leq \frac{\sqrt{2}}{\cosh(x)} + \left(\frac{\sqrt{2}}{\cosh(x)}\right)^3 \leq 2\sqrt{2}\,\mathrm{e}^{-|x|} + 16\sqrt{2}\,\mathrm{e}^{-3|x|} \\ &\leq 18\sqrt{2}\,\mathrm{e}^{-|x|}. \end{aligned}$$

Since $\mathrm{e}^{-|\cdot|} \in L^q(\mathbb{R})$, we conclude that $u_1 \in W^{2,q}(\mathbb{R})$ as claimed.

(b) This has been shown in the course of (a), see $(\diamondsuit)$.

(c) *First Step: We construct a branch of nontrivial solutions (u_λ, λ) $(\lambda > 0)$ by scaling.*

For $\lambda > 0$ and $x \in \mathbb{R}$, we define

$$u_\lambda(x) \coloneqq \sqrt{\lambda} \cdot u_1(\sqrt{\lambda}\,x) = \frac{\sqrt{2\lambda}}{\cosh(\sqrt{\lambda}\,x)} \qquad (x \in \mathbb{R}, \lambda > 0).$$

As in the first step, we see that $u_\lambda \in W^{2,q}(\mathbb{R})$ for every $q \in [1,\infty]$ and note that $u_\lambda \neq 0$. Moreover, u_λ is a smooth function and we have for $x \in \mathbb{R}$

$$-u_\lambda''(x) + \lambda u_\lambda(x) - u_\lambda^3(x) = \lambda^{\frac{3}{2}} \cdot \Big(- u_1''(\sqrt{\lambda}\,x) + u_1(\sqrt{\lambda}\,x) - u_1(\sqrt{\lambda}\,x)^3 \Big) = 0.$$

Hence, for all $\lambda > 0$, $u_\lambda \in W^{2,q}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ is a classical solution to

$$-u'' + \lambda u = u^3 \qquad \text{in } \mathbb{R}.$$

So we have found a family $\mathcal{T} := \{(u_\lambda, \lambda) : \ \lambda > 0\}$ of nontrivial solutions to (1).

<u>Second Step</u>: *We discuss whether $\mathcal{T}$ bifurcates from* $(0,0)$.

To this end, we fix $q \in [1, \infty]$, calculate the norms $\left\| u_\lambda^{(j)} \right\|_{L^q(\mathbb{R})}$, $j = 0, 1, 2$, and discuss the limit $\lambda \to 0$. We have for $\lambda > 0$ and $x \in \mathbb{R}$

$$u_\lambda(x) = \sqrt{\lambda} \cdot u_1(\sqrt{\lambda}\, x), \quad u_\lambda'(x) = \lambda \cdot u_1'(\sqrt{\lambda}\, x), \quad u_\lambda''(x) = \lambda\sqrt{\lambda} \cdot u_1''(\sqrt{\lambda}\, x).$$

Hence, for $j = 0, 1, 2$,

$$\left\| u_\lambda^{(j)} \right\|_{L^\infty(\mathbb{R})} = \lambda^{\frac{j+1}{2}} \left\| u_1^{(j)} \right\|_{L^\infty(\mathbb{R})} \overset{\lambda \to 0}{\longrightarrow} 0,$$

and for $1 \le q < \infty$,

$$\begin{aligned}\left\| u_\lambda^{(j)} \right\|_{L^q(\mathbb{R})}^q &= \int_{\mathbb{R}} |u_\lambda^{(j)}(x)|^q \,\mathrm{d}x = \int_{\mathbb{R}} |\lambda^{\frac{j+1}{2}} \cdot u_1^{(j)}(\sqrt{\lambda}\, x)|^q \,\mathrm{d}x \overset{y = \sqrt{\lambda}\, x}{=} \lambda^{\frac{qj+q-1}{2}} \int_{\mathbb{R}} |u_1^{(j)}(y)|^q \,\mathrm{d}y \\ &= \lambda^{\frac{qj+q-1}{2}} \left\| u_1^{(j)} \right\|_{L^q(\mathbb{R})}^q \overset{\lambda \to 0}{\longrightarrow} \begin{cases} 0, & 1 < q < \infty \text{ or } j = 1, 2, \\ \|u_1\|_{L^q(\mathbb{R})}^q > 0, & q = 1 \text{ and } j = 0. \end{cases}\end{aligned}$$

We conclude that

$$\lim_{\lambda \searrow 0} \|u_\lambda\|_{W^{2,q}(\mathbb{R})} = 0 \qquad \Longleftrightarrow \qquad 1 < q \le \infty,$$

and bifurcation from the trivial branch with respect to $\| \cdot \|_{W^{2,q}(\mathbb{R})}$ occurs if and only if $q \in (1, \infty]$.

$\square$

Problem 3:
Consider the boundary value problem

$$(2) \qquad \begin{cases} u'' + \lambda(u - u^3) = 0 & \text{in } (0, \pi), \\ u(0) = u(\pi) = 0. \end{cases}$$

Using the Energy Method, prove that for $\alpha \in (0,1)$ there exists a nontrivial positive solution $(u_\alpha, \lambda_\alpha)$ of (2) with $\|u_\alpha\|_\infty = \alpha$. Furthermore show that $(u_\alpha, \lambda_\alpha)$ bifurcates (with respect to $\| \cdot \|_\infty$ norm) from the trivial branch at $\lambda_0 = 1$.

Solution to problem 3:

<u>*Proof:*</u> We intend to apply Theorem 2.3 and Collorary 2.5 and thus define

$$g \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, g(z, \lambda) := \lambda(z - z^3).$$

Then $g \in C^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g(z, \lambda) = -g(-z, \lambda)$. Setting $\alpha_0 := 1$, we have $g(z, \lambda) > 0$ for $0 < z < 1$ and $\lambda > 0$.

Remark. In assumption (A) from the lecture, we required positivity for all $\lambda \in \mathbb{R}$. This was done to reduce notation, but statements from chapter 2 also hold true if we consider $\lambda \in \Lambda$ instead of $\lambda \in \mathbb{R}$, where Λ is some open interval. In fact, since any nonempty open interval Λ is homeomorphic to $\mathbb{R}$, by relabeling of λ we can always obtain a problem which is defined for $\lambda \in \mathbb{R}$.

Here, we consider $\Lambda := (0, \infty)$.

First, we show that for each $\alpha \in (0,1)$, a unique positive solution u with $\|u\|_\infty = \alpha$ of (2) exists. Let $G(z,\lambda) \coloneqq \int_0^z g(s,\lambda)\,\mathrm{d}s = \lambda(\frac{1}{2}u^2 - \frac{1}{4}u^4)$. For $\alpha \in (0,1)$ we calculate

$$\frac{T}{\sqrt{2}} \overset{!}{=} \int_0^\alpha (G(\alpha,\lambda) - G(z,\lambda))^{-1/2}\,\mathrm{d}z = \frac{2}{\sqrt{\lambda}} \int_0^\alpha \left(2\alpha^2 - \alpha^4 - 2z^2 + z^4\right)^{-1/2}\,\mathrm{d}z \tag{3}$$

Finiteness of this integral follows from

$$2\alpha^2 - \alpha^4 - 2z^2 + z^4 = 2(\alpha^2 - z^2)\cdot(1 - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}z^2) \geq 2(\alpha^2 - z^2)\cdot(1-\alpha^2).$$

combined with

$$\int_0^\alpha \frac{1}{\sqrt{\alpha^2 - z^2}}\,\mathrm{d}z = \frac{\pi}{2} < \infty$$

This shows that (3) has a solution $\lambda_\alpha > 0$. From Theorem 2.3 it follows that for each $\alpha \in (0,1)$, (2) has a positive solution u_α with norm $\|u_\alpha\|_\infty = \alpha$.

Lastly, since $g_z(0,1) = 1 = \frac{\pi^2(j+1)^2}{T^2}$ (since $T = 1$ and $j = 0$ here) and $g_{z\lambda}(0,1) = 1 \neq 0$, Collorary 2.5 shows that $(u_\alpha, \lambda_\alpha)$ bifurcate from the trivial branch at $(0,1)$. □