Institute for Analysis
PD Dr. Rainer Mandel
M.Sc. Sebastian Ohrem

Problem Sheet 2

Bifurcation Theory

Winter Semester 2022/23

7.11.2022

We consider the problem

$$\begin{cases} u''(t) + g(u(t), \lambda) = 0 \text{ for } t \in (0, T), \\ u(0) = u(T) = 0. \end{cases} \tag{1}$$

where

(A) $\alpha_0 \in (0, \infty]$ and $g \in C^1((-\alpha_0, \alpha_0) \times \mathbb{R}, \mathbb{R})$ with $g(z, \lambda) = -g(-z, \lambda) > 0$ for $0 < z < \alpha_0$, $\lambda \in \mathbb{R}$.

Recall the necessary and sufficient conditions appearing in Corollary 2.5 from the lecture:

(N) $g_z(0, \lambda_\star) = \frac{\pi^2 (j+1)^2}{T^2}$.

(S) $g_z(0, \cdot\,)$ is strictly monotone near $\lambda_\star$.

Problem 4:
Assume (A), (N) and the following stronger assumption, replacing (S):

(S') $g \in C^2((-\alpha_0, \alpha_0) \times \mathbb{R}, \mathbb{R})$ and $g_{z\lambda}(0, \lambda_\star) \neq 0$.

By Corollary 2.5, for $\alpha > 0$ sufficiently small there exist j-nodal solutions $(\pm u_\alpha, \lambda_\alpha)$ of (1) with $\|u\|_\infty = \alpha$ that bifurcate from $(0, \lambda_\star)$ w.r.t. $\|\cdot\|_\infty$.

(a) Prove that if (S') holds, then for α sufficiently small these solutions are uniquely determined by α.

(b) Prove that if (S') holds and $g_{zz}(0, \lambda) \neq 0$, then for α sufficiently smallthe bifurcation curve has the following "direction":
- λ_α is decreasing in α if $g_{zz}(0, \lambda_\star) g_{z\lambda}(0, \lambda_\star) > 0$,
- λ_α is increasing in α if $g_{zz}(0, \lambda_\star) g_{z\lambda}(0, \lambda_\star) < 0$.

Problem 5:
Assume that (A) and (N) hold, but (S) does not. Prove the following:

(a) Multiple bifurcation curves can exist at $(0, \lambda_\star)$, i.e. there exist j-nodal solutions $(u_\alpha, \mu_\alpha), (v_\alpha, \nu_\alpha)$ that bifurcate from $(0, \lambda_\star)$ such that $\|u_\alpha\|_\infty = \alpha = \|v_\alpha\|_\infty$ and $\mu_\alpha \neq \nu_\alpha$ for all α.

Remark: This does not show that the solutions $(u_\alpha, \mu_\alpha), (v_\alpha, \nu_\alpha)$ describe curves (i.e. that the maps $\alpha \mapsto (u_\alpha, \mu_\alpha)$, $\alpha \mapsto (v_\alpha, \nu_\alpha)$ are continuous). You need not show continuity.

(b) Bifurcation need not occur at $(0, \lambda_\star)$.

Hint: Consider $g(x, \lambda) = f(\lambda) \sin(x)$ for suitable f.

Problem 6:
Let $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $g(0, \lambda) = 0$ ($\lambda \in \mathbb{R}$) and $b \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $b(x, \lambda) \neq 0$ for all $x, \lambda \in \mathbb{R}$. Show that

$$u'' + b(x, \lambda) u' + g(u, \lambda) = 0 \tag{2}$$

does not admit nonconstant periodic solutions.