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Solution to Problem Sheet 2

Bifurcation Theory

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We consider the problem

$$
\begin{cases} u''(t) + g(u(t),\lambda) = 0 \text{ for } t \in (0,T), \\ u(0) = u(T) = 0. \end{cases} \tag{1}
$$

where

(A) $\alpha_0 \in (0,\infty]$ and $g \in C^1((-\alpha_0,\alpha_0)\times\mathbb{R},\mathbb{R})$ with $g(z,\lambda) = -g(-z,\lambda) > 0$ for $0 < z < \alpha_0$, $\lambda \in \mathbb{R}$.

Recall the necessary and sufficient conditions appearing in Corollary 2.5 from the lecture:

(N) $g_z(0,\lambda_\star) = \frac{\pi^2(j+1)^2}{T^2}$.

(S) $g_z(0,\,\cdot\,)$ is strictly monotone near $\lambda_\star$.

Problem 4:
Assume (A), (N) and the following stronger assumption, replacing (S):

(S') $g \in C^2((-\alpha_0,\alpha_0)\times\mathbb{R},\mathbb{R})$ and $g_{z\lambda}(0,\lambda_\star) \neq 0$.

By Corollary 2.5, for $\alpha > 0$ sufficiently small there exist j-nodal solutions $(\pm u_\alpha,\lambda_\alpha)$ of (1) with $\|u\|_\infty = \alpha$ that bifurcate from $(0,\lambda_\star)$ w.r.t. $\|\cdot\|_\infty$.

(a) Prove that if (S') holds, then for α sufficiently small these solutions are uniquely determined by α.

(b) Prove that if (S') holds and $g_{zz}(0,\lambda) \neq 0$, then for α sufficiently smallthe bifurcation curve has the following "direction":
- λ_α is decreasing in α if $g_{zz}(0,\lambda_\star)g_{z\lambda}(0,\lambda_\star) > 0$,
- λ_α is increasing in α if $g_{zz}(0,\lambda_\star)g_{z\lambda}(0,\lambda_\star) < 0$.

Solution to problem 4:
We revisit the proof of Corollary 2.5, and consider

$$
\frac{T}{\sqrt{2}(j+1)} = \int_0^\alpha \frac{1}{\sqrt{G(\alpha,\lambda)-G(z,\lambda)}}\,\mathrm{d}z = \int_0^1 \left(\frac{G(\alpha,\lambda)-G(s\alpha,\lambda)}{\alpha^2}\right)^{-1/2} \mathrm{d}s =: f(\alpha,\lambda) \tag{2.5}
$$

where $G(z,\lambda) = \int_0^z g(s,\lambda)\,\mathrm{d}s$ and we may write

$$
\begin{aligned}
\frac{G(\alpha,\lambda)-G(\alpha s,\lambda)}{\alpha^2} &= \int_s^1 \int_0^\tau g_z(\mu\alpha,\lambda)\,\mathrm{d}\mu\,\mathrm{d}\tau \\
&= \int_s^1 \int_0^\tau g_z(0,\lambda_0) + \mathrm{o}(1)\,\mathrm{d}\mu\,\mathrm{d}\tau = [g_z(0,\lambda_0)+\mathrm{o}(1)]\frac{1-s^2}{2}
\end{aligned} \tag{2}
$$

as $(\alpha,\lambda)\to(0,\lambda_0)$. Using dominated convergence, it follows that

$$
f(\alpha,\lambda) = \int_0^1 \left(\frac{G(\alpha,\lambda)-G(s\alpha,\lambda)}{\alpha^2}\right)^{-1/2} \mathrm{d}s \to \int_0^1 \left(g_z(0,\lambda_0)\frac{1-s^2}{2}\right)^{-1/2} \mathrm{d}s = \frac{1}{\sqrt{g_z(0,\lambda_0)}}\frac{\pi}{\sqrt{2}}.
$$

as $(\alpha,\lambda)\to(0,\lambda_0)$. Recall Theorem 2.3:

> There exists a j-nodal solution (u,λ) of (1) with $\|u\|_\infty = \alpha$ if and only if $f(\alpha,\lambda) = \frac{T}{\sqrt{2}(j+1)}$.

(a) We choose $\varepsilon, \delta > 0$ such that $g_{z\lambda} \neq 0$ on $(-\delta, \delta) \times (\lambda_\star - \varepsilon, \lambda_\star + \varepsilon)$. W.l.o.g. let $g_{z\lambda} > 0$ (otherwise replace λ by $-\lambda$).

As g_z is strictly increasing in λ, using (4) and the definition of f, we see that f is strictly decreasing in λ. Since in addition

$$f(0+, \lambda_\star - \varepsilon) = \frac{1}{\sqrt{g_z(0, \lambda_\star - \varepsilon)}} \frac{\pi}{\sqrt{2}} > \frac{1}{\sqrt{g_z(0, \lambda_\star)}} \frac{\pi}{\sqrt{2}} = \frac{T}{\sqrt{2}(j+1)} > \frac{1}{\sqrt{g_z(0, \lambda_\star + \varepsilon)}} \frac{\pi}{\sqrt{2}} = f(0+, \lambda_\star - \varepsilon),$$

for $\alpha > 0$ sufficiently small there exists a unique solution $\lambda \in (\lambda_\star - \varepsilon, \lambda_\star + \varepsilon)$ of $f(\alpha, \lambda) = \frac{T}{\sqrt{2}(j+1)}$.

By Theorem 2.3 this completes the proof.

(b) Again choose $\varepsilon, \delta > 0$ such that $g_{z\lambda}, g_{zz} \neq 0$ on $(-\delta, \delta) \times (\lambda_\star - \varepsilon, \lambda_\star + \varepsilon)$. W.l.o.g. let $g_{z\lambda} > 0$. Also we only consider the case $g_{zz} > 0$, as $g_{zz} < 0$ can be treated similarly.

Problem 5:
Assume that (A) and (N) hold, but (S) does not. Prove the following:

(a) Multiple bifurcation curves can exist at $(0, \lambda_\star)$, i.e. there exist j-nodal solutions $(u_\alpha, \mu_\alpha), (v_\alpha, \nu_\alpha)$ that bifurcate from $(0, \lambda_\star)$ such that $\|u_\alpha\|_\infty = \alpha = \|v_\alpha\|_\infty$ and $\mu_\alpha \neq \nu_\alpha$ for all α.

Remark: This does not show that the solutions $(u_\alpha, \mu_\alpha), (v_\alpha, \nu_\alpha)$ describe curves (i.e. that the maps $\alpha \mapsto (u_\alpha, \mu_\alpha)$, $\alpha \mapsto (v_\alpha, \nu_\alpha)$ are continuous). You need not show continuity.

(b) Bifurcation need not occur at $(0, \lambda_\star)$.

Hint: Consider $g(x, \lambda) = f(\lambda) \sin(x)$ for suitable f.

Solution to problem 5:

(a) Consider, as a slight modification of the pendulum equation discussed in the lecture, $g(z, \mu) = (\lambda_j + (\mu - \lambda_j)^2) \sin(z)$ where $\lambda_j = ((j+1)\pi)^2$ for some $j \in \mathbb{N}_0$, and consider $\mu_\star := \lambda_j$.

We know from the lecture that, for $-u'' = \lambda \sin(u), u(0) = u(1) = 0$, j-nodal solutions bifurcate at the point $(0, \lambda_j)$, parametrized as $(u_{\alpha,j}, \lambda_{\alpha,j})_{0<\alpha<\pi}$ with $\lambda_j(\alpha) \searrow \lambda_j$ as $\alpha \searrow 0$. With g chosen as above, we find two parameters μ corresponding to each value of α via

$$\lambda_{\alpha,j} = \lambda_j + (\mu - \lambda_j)^2 \quad \iff \quad \mu = \lambda_j \pm \sqrt{\lambda_{\alpha,j} - \lambda_j},$$

which yields two distinct families of bifurcating j-nodal solutions of (1) parametrized as

$$\left(u_{\alpha,j}, \lambda_j + \sqrt{\lambda_j(\alpha) - \lambda_j}\right)_{0<\alpha<\pi}, \quad \left(u_{\alpha,j}, \lambda_j - \sqrt{\lambda_j(\alpha) - \lambda_j}\right)_{0<\alpha<\pi}.$$

(b) Again, we modify the pendulum equation. We introduce $g(z, \mu) = (\lambda_j - (\mu - \lambda_j)^2) \sin(z)$ where $\lambda_j = ((j+1)\pi)^2$ for some $j \in \mathbb{N}_0$, and consider $\mu_\star := \lambda_j$.

However, using the notation from part (a), the equation

$$\lambda_{\alpha,j} = \lambda_j - (\mu - \lambda_j)^2$$

does not have solutions, and the bifurcation diagram for $-u'' = \lambda \sin(u), u(0) = u(1) = 0$ from the lecture reveals that there is no bifurcation of (1) at $(0, \lambda_j)$.

Problem 6:
Let $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $g(0, \lambda) = 0$ $(\lambda \in \mathbb{R})$ and $b \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $b(x, \lambda) \neq 0$ for all $x, \lambda \in \mathbb{R}$. Show that

$$u'' + b(x, \lambda) u' + g(u, \lambda) = 0 \tag{3}$$

does not admit nonconstant periodic solutions.

Solution to problem 6:
First, we note that by assumption, b is continuous and does not have any zero, so b is either negative or positive on all of $\mathbb{R} \times \mathbb{R}$.

We assume that $u \in C^2(\mathbb{R})$ is a periodic solution of (3). We introduce

$$E : \mathbb{R} \to \mathbb{R}, \quad E(t) \coloneqq \frac{1}{2}[u'(t)]^2 + G(u(t), \lambda)$$

where $G(z, \lambda) \coloneqq \int_0^z g(s, \lambda)\, \mathrm{d}s$ for $\lambda, z \in \mathbb{R}$. Then, $E \in C^1(\mathbb{R})$, and for $t \in \mathbb{R}$

$$E'(t) = u'(t) \cdot (u''(t) + g(u(t), \lambda)) = -b(t, \lambda)[u'(t)]^2 \tag{4}$$

where we have inserted the differential equation in the last step. Since b does not change sign, this implies that E is a monotone function. As u is periodic, so is E, and we conclude that E is constant.

Hence, $E' \equiv 0$, and by (4), $u' \equiv 0$. So u is constant, and the assertion is proved.