Karlsruhe Institute of Technology

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Problem Sheet 3

Bifurcation Theory

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Problem 7:
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\varphi \in C^1(\overline{\Omega} \times \mathbb{R})$. We consider the Banach space $C(\overline{\Omega})$ endowed with the norm $\|u\|_\infty := \max_{x \in \overline{\Omega}} |u(x)|$, $u \in C(\overline{\Omega})$. Prove that the map

$$F : C(\overline{\Omega}) \to C(\overline{\Omega}), \quad (F(u))(x) := \varphi(x, u(x)) \qquad (x \in \overline{\Omega})$$

is continuously Fréchet differentiable and calculate its derivative.

Problem 8:
Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ an open subset, $2 \leq p < \frac{2n}{n-2}$ for $n \geq 3$ and $2 \leq p < \infty$ else. Prove that the map

$$F : H_0^1(\Omega) \to \mathbb{R}, \quad F(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p} \int_\Omega |u|^p \,\mathrm{d}x$$

is continuously Fréchet differentiable and calculate the derivative.

Hint: By choice of p, the continuous Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ holds, i.e. there exists $C > 0$ with the property that $\|u\|_{L^p(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}$ for every $u \in H_0^1(\Omega)$.

Problem 9:
On the Banach space $L^\infty(\mathbb{R})$, we consider the map $F : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, $F(u) := |u|^{\frac{1}{2}}$. Let $u_0 := \mathbb{1}_{[0,1]} \in L^\infty(\mathbb{R})$. For $h \in L^\infty(\mathbb{R})$, prove that the directional derivative

$$\lim_{t \to 0} \frac{F(u_0 + th) - F(u_0)}{t}$$

exists if and only if $h(x) = 0$ for almost all $x \in \mathbb{R} \setminus [0, 1]$. Conclude that F is not Gâteaux differentiable in the point u_0.