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Solution to Problem Sheet 3

Bifurcation Theory

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Problem 7:
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\varphi \in C^1(\overline{\Omega} \times \mathbb{R})$. We consider the Banach space $C(\overline{\Omega})$ endowed with the norm $\|u\|_\infty := \max_{x\in\overline{\Omega}} |u(x)|$, $u \in C(\overline{\Omega})$. Prove that the map

$$F : C(\overline{\Omega}) \to C(\overline{\Omega}), \quad F(u)(x) := \varphi(x, u(x)) \qquad (x \in \overline{\Omega})$$

is continuously Fréchet differentiable and calculate its derivative.

Solution to problem 7:

Proof: **Step 1:** Gâteaux differentiability of F.

Let $u, h \in C(\overline{\Omega})$. We have to show that the limit

$$\lim_{t\to 0} \frac{F(u+th) - F(u)}{t}$$

exists in $C(\overline{\Omega})$. First, we prove existence of a pointwise limit. So we fix $x \in \overline{\Omega}$ and calculate

$$\begin{aligned}\lim_{t\to 0} \frac{F(u+th)(x) - F(u)(x)}{t} &= \lim_{t\to 0} \frac{\varphi(x, u(x) + th(x)) - \varphi(x, u(x))}{t} \\ &= \frac{\partial\varphi}{\partial x_{n+1}}(x, u(x)) \cdot h(x) =: \varphi_u(x, u(x)) \cdot h(x)\end{aligned}$$

Hence, the only candidate for a Gâteaux derivative is $(\mathrm{d}F(u)[h])(x) = \varphi_u(x, u(x)) \cdot h(x)$.

Next we have to show that this limit is uniform (i.e. that we have convergence in the Banach space $C(\overline{\Omega})$). Let $\varepsilon > 0$. Since φ_u is uniformly continuous on the compact set $\overline{\Omega} \times [-M, M]$, $M := \|u\|_\infty + \|h\|_\infty$, we find $\delta > 0$ with

$$\forall x \in \overline{\Omega}\ \forall z_1, z_2 \in [-M, M] \colon |z_1 - z_2| < \delta \implies |\varphi_u(x, z_1) - \varphi_u(x, z_2)| < \frac{\varepsilon}{\max\{1, \|h\|_\infty\}}.$$

So for all $\tilde{t} \in [-1, 1]$ with $|\tilde{t}|\|h\|_\infty < \delta$, we have

$$\forall x \in \overline{\Omega} : \qquad |\varphi_u(x, u(x) + \tilde{t}h(x)) - \varphi_u(x, u(x))| < \frac{\varepsilon}{\max\{1, \|h\|_\infty\}}$$

and find for $t \in [-1, 1]$ with $|t|\|h\|_\infty < \delta$, using the fundamental theorem of calculus,

$$\begin{aligned}&\max_{x\in\overline{\Omega}} \left| \frac{\varphi(x, u(x) + th(x)) - \varphi(x, u(x))}{t} - \varphi_u(x, u(x))h(x) \right| \\ &= \max_{x\in\overline{\Omega}} \left| \frac{1}{t} \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\sigma} \Big[\varphi(x, u(x) + \sigma th(x)) \Big] - t\varphi_u(x, u(x))h(x) \,\mathrm{d}\sigma \right) \right| \\ &= \max_{x\in\overline{\Omega}} \left| \int_0^1 \varphi_u(x, u(x) + \sigma th(x))h(x) - \varphi_u(x, u(x))h(x) \,\mathrm{d}\sigma \right| \\ &\le \max_{x\in\overline{\Omega}} \int_0^1 \big|\varphi_u(x, u(x) + \sigma th(x)) - \varphi_u(x, u(x))\big| |h(x)| \,\mathrm{d}\sigma\end{aligned}$$

$$< \max_{x\in\overline{\Omega}} \int_0^1 \frac{\varepsilon}{\max\{1, \|h\|_\infty\}} \|h\|_\infty \, \mathrm{d}\sigma = \frac{\varepsilon}{\max\{1, \|h\|_\infty\}} \|h\|_\infty \leq \varepsilon.$$

This shows that F is Gâteaux differentiable at u with

$$(dF(u)[h])(x) = \varphi_u(x, u(x))h(x).$$

Step 2: Continuity of the Gâteaux derivative of F.

We need to show that the map

$$C(\overline{\Omega}) \to \mathcal{L}(C(\overline{\Omega}); C(\overline{\Omega})), u \mapsto dF(u)$$

is well-defined and continuous. First for $h \in C(\overline{\Omega})$ we estimate

$$\|dF(u)[h]\|_\infty = \sup_{x\in\overline{\Omega}} |\varphi_u(x, u(x))h(x)| \leq \max_{x\in\overline{\Omega}} |\varphi_u(x, u(x))| \|h\|_\infty$$

where the Maximum exists since $\varphi_u(\,\cdot\,, u(\,\cdot\,))$ is continuous and $\overline{\Omega}$ is compact. Clearly, $dF(u)$ is linear in h, which shows that

$$dF(u)[h] \in \mathcal{L}(C(\overline{\Omega}); C(\overline{\Omega}))$$

for each $u \in C(\overline{\Omega})$.

It remains to show continuity of dF. To do this, consider functions $u_n, u \in C(\overline{\Omega})$, $n \in \mathbb{N}$, with $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. We have to show $dF(u_n) \to dF(u)$ as $n \to \infty$ in $\mathcal{L}(C(\overline{\Omega}), C(\overline{\Omega}))$, i.e.

$$\sup_{h\in C(\overline{\Omega}),\, \|h\|_\infty = 1} \|dF(u_n)[h] - dF(u)[h]\|_\infty \to 0.$$

We let $\varepsilon > 0$ and, using uniform continuity of φ_u in a similar way as in the previous part, we find $\delta > 0$ with

$$\forall x \in \overline{\Omega}\ \forall z_1, z_2 \in [-M_1, M_1] \colon |z_1 - z_2| < \delta \implies |\varphi_u(x, z_1) - \varphi_u(x, z_2)| < \frac{\varepsilon}{\max\{1, \|h\|_\infty\}}.$$

where $M_1 := \|u\|_\infty + 1$. As $u_n \to u$ uniformly on $\overline{\Omega}$, we also find $n_0 \in \mathbb{N}$ with $\|u_n\|_\infty \leq M_1$ and $|u_n(x) - u(x)| < \delta$ for all $n \geq n_0$ and $x \in \overline{\Omega}$. We estimate for $n \geq n_0$

$$\begin{aligned}
&\sup_{h\in C(\overline{\Omega}),\, \|h\|_\infty = 1} \|dF(u_n)[h] - dF(u)[h]\|_\infty \\
&\quad\leq \sup_{h\in C(\overline{\Omega}),\, \|h\|_\infty = 1} \left(\max_{x\in\overline{\Omega}} |\varphi_u(x, u_n(x)) - \varphi_u(x, u(x))| \cdot \|h\|_\infty \right) \\
&\quad= \max_{x\in\overline{\Omega}} |\varphi_u(x, u_n(x)) - \varphi_u(x, u(x))| < \varepsilon,
\end{aligned}$$

which shows continuity of the Gâteaux derivative. From the lecture, Short exercises after Definition 3.2 we can conclude that F is continuously Fréchet differentiable with $F'(u)[h] = dF(u)[h]$. ☐

Problem 8:
Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ an open subset, $2 \leq p < \frac{2n}{n-2}$ for $n \geq 3$ and $2 \leq p < \infty$ else. Prove that the map

$$F : H_0^1(\Omega) \to \mathbb{R}, \quad F(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \frac{1}{p} \int_\Omega |u|^p \, \mathrm{d}x$$

is continuously Fréchet differentiable and calculate the derivative.

Hint: By choice of p, the continuous Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ holds, i.e. there exists $C > 0$ with the property that $\|u\|_{L^p(\Omega)} \leq C \|u\|_{H_0^1(\Omega)}$ for every $u \in H_0^1(\Omega)$.

Solution to problem 8:
Claim: F is continuously Fréchet differentiable with

$$F'(u)[h] = \int_\Omega \nabla u \cdot \nabla h - |u|^{p-2} uh \,\mathrm{d}x$$

for $u, h \in H_0^1(\Omega)$.

Proof: We discuss seperately $F_0, F_1 : H_0^1(\Omega) \to \mathbb{R}$,

$$F_0(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \qquad F_1(u) := \frac{1}{p} \int_\Omega |u|^p \,\mathrm{d}x.$$

Continuous Fréchet differentiability of F_0:

For $u, h \in H_0^1(\Omega)$, we calculate directly:

$$\begin{aligned} F_0(u+h) - F_0(u) &= \frac{1}{2} \int_\Omega \nabla(u+h) \cdot \nabla(u+h) - \nabla u \cdot \nabla u \,\mathrm{d}x \\ &= \int_\Omega \nabla u \cdot \nabla h + \frac{1}{2} |\nabla h|^2 \,\mathrm{d}x \end{aligned}$$

and hence, using the norm $\|h\|_{H^1(\Omega)}^2 := \int_\Omega |\nabla h|^2 + h^2 \,\mathrm{d}x$,

$$\left| F_0(u+h) - F_0(u) - \int_\Omega \nabla u \cdot \nabla h \,\mathrm{d}x \right| = \frac{1}{2} \int_\Omega |\nabla h|^2 \,\mathrm{d}x \leq \frac{1}{2} \|h\|_{H^1(\Omega)}^2 = o(\|h\|_{H^1(\Omega)});$$

moreover, $H_0^1(\Omega) \to \mathbb{R}, \ h \mapsto \int_\Omega \nabla u \cdot \nabla h \,\mathrm{d}x$ is a continuous linear operator. We conclude that F_0 is Fréchet differentiable on $H_0^1(\Omega)$ with $F_0'(u)[h] = \int_\Omega \nabla u \cdot \nabla h \,\mathrm{d}x$ for $u, h \in H_0^1(\Omega)$. To see (even Lipschitz) continuity of F_0', we estimate for $u, v \in H_0^1(\Omega)$

$$\begin{aligned} \|F_0'(u) - F_0'(v)\|_{\mathcal{L}(H_0^1(\Omega), \mathbb{R})} &= \sup_{h \in H_0^1(\Omega), \, \|h\|_{H^1(\Omega)} = 1} |F_0'(u)[h] - F_0'(v)[h]| \\ &= \sup_{h \in H_0^1(\Omega), \, \|h\|_{H^1(\Omega)} = 1} \int_\Omega |\nabla(u-v) \cdot \nabla h| \,\mathrm{d}x \\ &\leq \sup_{h \in H_0^1(\Omega), \, \|h\|_{H^1(\Omega)} = 1} \big(\|u-v\|_{H^1(\Omega)} \|h\|_{H^1(\Omega)} \big) = \|u-v\|_{H^1(\Omega)}. \end{aligned}$$

Continuous Fréchet differentiability of F_1:

We proceed as in Problem 8 and first show Gâteaux differentiability of F_1 and then linearity, boundedness and continuity of the Gâteaux derivative.

Step 1: Gâteaux differentiability of F_1.

Let $u, h \in H_0^1(\Omega)$. By definition of Gâteaux differentiability, we have to prove that

$$\mathbb{R} \to \mathbb{R}, \quad t \mapsto F_1(u+th) = \frac{1}{p} \int_\Omega |u+th|^p \,\mathrm{d}x$$

is differentiable at $t = 0$. This will be achieved via a standard consequence of dominated convergence (differentiation under the integral sign).

We let $f(t,x) := \frac{1}{p} |u(x) + th(x)|^p$ for $x \in \Omega, t \in \mathbb{R}$. Then, for $x \in \Omega$, f is differentiable w.r.t. t, and $\frac{\partial f}{\partial t}(t,x) = |u(x) + th(x)|^{p-2}(u(x) + th(x))h(x)$. Moreover, for $-1 < t < 1$, this expression is majorized by

$$\left| \frac{\partial f}{\partial t}(t,x) \right| \leq |u(x) + th(x)|^{p-1} |h(x)| \leq (|u(x)| + |h(x)|)^p \leq 2^p (|u(x)|^p + |h(x)|^p),$$

which is integrable due to the Sobolev embedding given in the hint. Dominated convergence now gives, for $-1 < t < 1$, existence of the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(F_1(u+th)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f(t,x)\,\mathrm{d}x = \int_\Omega \frac{\partial f}{\partial t}(t,x)\,\mathrm{d}x = \int_\Omega |u+th|^{p-2}(u+th)h\,\mathrm{d}x.$$

Thus we have

$$dF(u)[h] = \int_\Omega |u|^{p-2}uh\,\mathrm{d}x$$

for $u, h \in H_0^1(\Omega)$

By the continuous Sobolev embedding in the hint and Hölder's inequality, we see that $dF(u)\colon H_0^1(\Omega) \to \mathbb{R}$ is linear and bounded.

Step 2: Continuity of the Gâteaux derivative of F_1.

Let $u_n, u \in H_0^1(\Omega)$ with $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$. By the Riesz-Fischer Theorem (or a proof thereof) there exists a subsequence (u_{n_k}) that converges pointwise a.e. to u and there exists some $h \in L^p(\Omega)$ such that $|u_{n_k}| \leq h$ for all $k \in \mathbb{N}$. Now for $k \in \mathbb{N}$ we estimate

$$\begin{aligned}
&\|dF_1(u_{n_k}) - dF_1(u)\| \\
&\quad= \sup_{\|h\|_{H_0^1(\Omega)}\leq 1} |dF_1(u_{n_k})[h] - dF_1(u)[h]| \\
&\quad= \sup_{\|h\|_{H_0^1(\Omega)}\leq 1} \left|\int_\Omega |u_{n_k}|^{p-2}u_{n_k}h - |u|^{p-2}uh\,\mathrm{d}x\right| \\
&\quad\leq \sup_{\|h\|_{H_0^1(\Omega)}\leq 1} \Big\||u_{n_k}|^{p-2}u_{n_k} - |u|^{p-2}u\Big\|_{p'}\|h\|_p \\
&\quad\leq C\Big\||u_{n_k}|^{p-2}u_{n_k} - |u|^{p-2}u\Big\|_{p'} \\
&\quad= C\left(\int_\Omega \Big||u_{n_k}|^{p-2}u_{n_k} - |u|^{p-2}u\Big|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}
\end{aligned}$$

with $p' = \frac{p}{p-1}$, i.e. $\frac{1}{p'} + \frac{1}{p} = 1$, and where C is the constant appearing in the Sobolev Embedding. The integrand can be bounded from above via

$$\Big||u_{n_k}|^{p-2}u_{n_k} - |u|^{p-2}u\Big|^{p'} \leq \big(2h^{p-1}\big)^{p'} = 2^{p'}h^p$$

so that by dominated convergence we have

$$\int_\Omega \Big||u_{n_k}|^{p-2}u_{n_k} - |u|^{p-2}u\Big|^{p'}\,\mathrm{d}x \to 0$$

as $k \to \infty$, and thus $dF_1(u_{n_k}) \to dF_1(u)$ in $\mathcal{L}(H_0^1(\Omega);\mathbb{R})$ as $k \to \infty$. Replacing u_n by any subsequence in the above argument, we find that any subsequence u_{n_k} contains another subsequence $u_{n_{k_j}}$ such that $F_1(u_{n_{k_j}}) \to F_1(u)$ as $j \to \infty$. Thus $F_1(u_n) \to F_1(u)$ as $n \to \infty$. We conclude that F_1 is continuously Fréchet differentiable and

$$F_1'(u)[h] = dF_1(u)[h] = \int_\Omega |u|^{p-2}uh\,\mathrm{d}x.$$

In total, we have seen that $F = F_0 - F_1$ is Fréchet differentiable and its derivative is given by

$$F'(u)[h] = \int_\Omega \nabla u\cdot\nabla h - |u|^{p-2}uh\,\mathrm{d}x. \qquad \square$$

Problem 9:
On the Banach space $L^\infty(\mathbb{R})$, we consider the map $F : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}),\ F(u) := |u|^{\frac{1}{2}}$. Let $u_0 := \mathbb{1}_{[0,1]} \in L^\infty(\mathbb{R})$. For $h \in L^\infty(\mathbb{R})$, prove that the directional derivative

$$\lim_{t\to 0} \frac{F(u_0 + th) - F(u_0)}{t}$$

exists if and only if $h(x) = 0$ for almost all $x \in \mathbb{R} \setminus [0,1]$. Conclude that F is not Gâteaux differentiable in the point u_0.

Solution to problem 9:

Proof: (i) First, we consider $h \in L^\infty(\mathbb{R})$ which does not satisfy $h(x) = 0$ a.e. on $\mathbb{R} \setminus [0,1]$.

By assumption, there exists such $\delta > 0$ that the set $N := \{|h| \geq \delta\} \setminus [0,1]$ has positive measure. Then for almost every $x \in N$, we have $u_0(x) = 0$ and $|h(x)| \geq \delta$ and estimate as follows for $t \neq 0$:

$$\left|\frac{(F(u_0 + th))(x) - (F(u_0))(x)}{t}\right| = \left|\frac{|u_0(x) + th(x)|^{\frac{1}{2}} - |u_0(x)|^{\frac{1}{2}}}{t}\right| = \left|\frac{h(x)}{t}\right|^{\frac{1}{2}} \geq \sqrt{\delta}|t|^{-\frac{1}{2}}.$$

Thus, as $t \to 0$,

$$\left\|\frac{F(u_0 + th) - F(u_0)}{t}\right\|_{L^\infty(\mathbb{R})} \geq \left\|\frac{F(u_0 + th) - F(u_0)}{t}\right\|_{L^\infty(N)} \geq \sqrt{\delta}|t|^{-\frac{1}{2}} \to \infty,$$

and hence the directional derivative in direction h does not exist.

(i) Now, we let $h \in L^\infty(\mathbb{R})$ with $h(x) = 0$ a.e. on $\mathbb{R}\setminus[0,1]$. We will prove that the directional derivative exists and is given by

$$\lim_{t\to 0} \frac{F(u_0 + th) - F(u_0)}{t} = \frac{1}{2}h.$$

The case $h = 0$ is trivial, hence we can assume $h \neq 0$ in the following. For $t \in \mathbb{R}$ with $0 < |t| < \frac{1}{2\|h\|_\infty}$, we have

$$\begin{aligned}\left\|\frac{F(u_0 + th) - F(u_0)}{t} - \frac{1}{2}h\right\|_{L^\infty(\mathbb{R})} &= \left\|\frac{|u_0 + th|^{\frac{1}{2}} - |u_0|^{\frac{1}{2}}}{t} - \frac{1}{2}h\right\|_{L^\infty([0,1])} \\ &= \operatorname*{ess\,sup}_{0\leq x\leq 1}\left|\frac{\sqrt{1 + th(x)} - 1 - \frac{1}{2}th(x)}{t}\right|.\end{aligned}$$

We estimate further by Taylor expansion. For almost every $x \in [0,1]$, we have $|th(x)| \leq \frac{1}{2}$, and there exists $\xi(x,t) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ with

$$\sqrt{1 + th(x)} = 1 + th(x) \cdot \frac{1}{2} + \frac{1}{2}t^2h(x)^2 \cdot \left(-\frac{1}{4}\right)(1 + \xi(x,t))^{-\frac{3}{2}},$$

hence we estimate the expression above

$$\begin{aligned}\left\|\frac{F(u_0 + th) - F(u_0)}{t} - \frac{1}{2}h\right\|_{L^\infty(\mathbb{R})} &= \operatorname*{ess\,sup}_{0\leq x\leq 1}\left|\frac{1}{t} \cdot \frac{1}{2}t^2h(x)^2 \cdot \left(-\frac{1}{4}\right)(1 + \xi(x,t))^{-\frac{3}{2}}\right| \\ &\leq \operatorname*{ess\,sup}_{0\leq x\leq 1}\left|\frac{1}{8}th(x)^2 \cdot (1 + \xi(x,t))^{-\frac{3}{2}}\right| \\ &\leq \frac{1}{8}t\|h\|^2_{L^\infty(\mathbb{R})} \cdot \left(\frac{1}{2}\right)^{-\frac{3}{2}} = 2^{-\frac{3}{2}}\|h\|^2_{L^\infty(\mathbb{R})} \cdot t.\end{aligned}$$

We conclude that, as asserted,

$$\left\|\frac{F(u_0 + th) - F(u_0)}{t} - \frac{1}{2}h\right\|_{L^\infty(\mathbb{R})} \to 0 \qquad \text{as} \quad t \to 0.$$

Finally, we infer that F is not Gâteaux differentiable at the point u_0. In fact, Gâteaux differentiability would in particular imply existence of all directional derivatives,

$$\lim_{t\to 0} \frac{F(u_0 + th) - F(u_0)}{t} = \mathrm{d}F(u_0)[h],$$

thereby contradicting part (i). □