Karlsruhe Institute of Technology

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Problem Sheet 4

Bifurcation Theory

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Problem 10:
For $\lambda \in \mathbb{R}$, we consider the boundary value problem

$$\begin{cases} u'' - \sin(u) = \lambda e^x & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

Show that there exists $\delta > 0$ such that (1) admits a solution $\hat{u}(\lambda) \in C^2([0,1])$ for $|\lambda| < \delta$.

Problem 11 (Continuation of simple eigenvalues):
Let $\lambda_0 \in \mathbb{R}$, $d \in \mathbb{N}$, and $A \in C^1(\mathbb{R}; \mathbb{C}^{d\times d})$. Assume that $\mu_0 \in \mathbb{C}$ is eigenvalue of $A(\lambda_0)$ of algebraic multiplicity 1.

(a) Show that there exist open sets $U \subseteq \mathbb{R}$, $V \subseteq \mathbb{C}$ with $\lambda_0 \in U$, $\mu_0 \in V$ and a map $\hat{\mu} \in C^1(U; V)$ with

$$\mu \text{ is eigenvalue of } A(\lambda) \iff \mu = \hat{\mu}(\lambda)$$

for all $\lambda \in U, \mu \in V$.

(b) Show that there exists an open $U_1 \subseteq U$ with $\lambda_0 \in U_1$ so that $\mu(\lambda)$ is of algebraic multiplicity 1 for $\lambda \in U_1$.

(c) Show that there exist an open $U_2 \subseteq U$ with $\lambda_0 \in U_2$ and a function $\hat{x} \in C^1(U_2; \mathbb{C}^d \setminus \{0\})$ such that $\ker(A(\lambda) - \hat{\mu}(\lambda)I) = \mathbb{C}\hat{x}(\lambda)$ for $\lambda \in U_2$.

Problem 12:
Let $\alpha \in (0,1)$, $\emptyset \neq \Omega \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\gamma \in C^{2,\alpha}(\partial\Omega)$. Consider the minimal surface boundary value problem:

$$\begin{cases} (1+u_y^2)u_{xx} + (1+u_x^2)u_{yy} - 2u_x u_y u_{xy} = 0 & \text{in } \Omega, \\ u = \gamma & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Prove that for $\gamma \in C^{2,\alpha}(\partial\Omega)$ sufficiently small there exists a solution $u \in C^{2,\alpha}(\overline{\Omega})$ of (2).

You may use without proof that for all $g \in C^\alpha(\overline{\Omega})$ and $h \in C^{2,\alpha}(\partial\Omega)$ the problem

$$\begin{cases} \Delta u = g & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ and that there exists $C > 0$ such that

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\Big(\|g\|_{C^\alpha(\overline{\Omega})} + \|h\|_{C^{2,\alpha}(\partial\Omega)}\Big).$$

For a proof, see *[D.Gilbarg, N.S.Trudinger, Elliptic Partial Differential Equations of Second Order, 1977]*, Theorem 6.14, the end of chapter 6.3, and Lemma 6.38.