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Solution to Problem Sheet 4

Bifurcation Theory

Winter Semester 2022/23

28.11.2022

Problem 10:
For $\lambda \in \mathbb{R}$, we consider the boundary value problem

$$\begin{cases} u'' - \sin(u) = \lambda e^x & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

Show that there exists $\delta > 0$ such that (1) admits a solution $\hat{u}(\lambda) \in C^2([0,1])$ for $|\lambda| < \delta$.

Solution to problem 10:
We consider the Banach spaces $Z \coloneqq C([0,1];\mathbb{R})$ with norm $\|u\|_{C([0,1])} \coloneqq \|u\|_\infty$, $u \in Z$, and $X \coloneqq \{u \in C^2([0,1],\mathbb{R}) : u(0) = u(1) = 0\}$ with norm $\|u\|_{C^2([0,1])} \coloneqq \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty$, $u \in X$.[1] Moreover, we define

$$F : X \times \mathbb{R} \to Z, \quad (F(u,\lambda))(x) \coloneqq u''(x) - \sin(u(x)) - \lambda e^x \quad (0 \le x \le 1).$$

Then $u \in C^2([0,1])$ solves (1) if and only if $u \in X$ and $F(u,\lambda) = 0$ hold.

In order to apply the Implicit Function Theorem in a neighborhood of the zero $(0,0)$ of F, we check differentiability of F.

<u>*Claim:*</u> F is Fréchet differentiable and

$$F'(u,\lambda)[(w,\mu)](x) = w''(x) - \cos(u(x))w(x) - \mu e^x.$$

To show this, we estimate

$$\begin{aligned} &\left\| F(u+w,\lambda+\mu) - F(u,\lambda) - \left(w'' - \cos(u)w - \mu e^{(\cdot)}\right) \right\|_Z \\ &= \|\sin(u+w) - \sin(u) - \cos(u)w\|_\infty \\ &= \left\| -\tfrac{1}{2}\sin(\xi)w^2 \right\|_\infty \\ &\le \tfrac{1}{2}\|w\|_\infty^2 \\ &\le \tfrac{1}{2}\|w\|_X^2 = o(\|w\|_X) \end{aligned}$$

as $w \to 0$. The second equality above (with $\xi(x)$ lying between $u(x)$ and $u(x) + w(x)$) is due to Taylor expansion.

To apply the IFT at $F(0,0) = 0$, it remains to show that $A \coloneqq F_u(0,0) \colon X \to Z$ given by $F_u(0,0)[w] = w'' - w$ is invertible.

- We show that A is injective: Let $w \in X$ with $Aw = w'' - w = 0$. The general solution of the ODE is given by $w = \alpha \sinh + \beta \cosh$, and from the boundary conditions we obtain

  $$0 = w(0) = \alpha, \qquad 0 = w(1) = \alpha\cosh(1) + \beta\sinh(1),$$

  hence $\alpha = \beta = 0$ and $w \equiv 0$.

[1] X is complete because it is a closed subset of the Banach space $C^2([0,1],\mathbb{R})$ with norm $\|\cdot\|_{C^2([0,1])}$.

- For boundary value problems of this type, injectivity and surjectivity are generally equivalent. This statement is called Fredholm alternative for boundary value problems.

  We give a self-contained proof of surjectivity in this case: Let $f \in Z = C([0,1])$ be arbitrary. Then the initial value problem

$$\begin{cases} v'' - v = f \text{ in } [0,1], \\ v(0) = v'(0) = 0 \end{cases}$$

  has a unique solution $v \in C^2([0,1])$ by the Picard-Lindelöf theorem. We immediately see that $w = v - \frac{v(1)}{\sinh(1)} \sinh$ lies in X and satisfies $Aw = f$.

- Continuity of the inverse map A^{-1} follows from the Bounded inverse theorem.

With all conditions of the IFT checked, we can apply it to obtain existence of a $\delta > 0$ and a smooth map $\hat{u} \in C^1((-\delta,\delta); Z)$ with $F(\hat{u}(\lambda), \lambda) = 0$ for all $\lambda \in (-\delta, \delta)$.

Problem 11 (Continuation of simple eigenvalues):
Let $\lambda_0 \in \mathbb{R}$, $d \in \mathbb{N}$, and $A \in C^1(\mathbb{R}; \mathbb{C}^{d\times d})$. Assume that $\mu_0 \in \mathbb{C}$ is eigenvalue of $A(\lambda_0)$ of algebraic multiplicity 1.

(a) Show that there exist open sets $U \subseteq \mathbb{R}$, $V \subseteq \mathbb{C}$ with $\lambda_0 \in U$, $\mu_0 \in V$ and a map $\hat{\mu} \in C^1(U; V)$ with

$$\mu \text{ is eigenvalue of } A(\lambda) \iff \mu = \hat{\mu}(\lambda)$$

for all $\lambda \in U, \mu \in V$.

(b) Show that there exists an open $U_1 \subseteq U$ with $\lambda_0 \in U_1$ so that $\mu(\lambda)$ is of algebraic multiplicity 1 for $\lambda \in U_1$.

(c) Show that there exist an open $U_2 \subseteq U$ with $\lambda_0 \in U_2$ and a function $\hat{x} \in C^1(U_2; \mathbb{C}^d \setminus \{0\})$ such that $\ker(A(\lambda) - \hat{\mu}(\lambda)I) = \mathbb{C}\hat{x}(\lambda)$ for $\lambda \in U_2$.

Solution to problem 11:
In the following, we consider the following generalization of the given problem:

Replacing $\mathbb{C}^d$, X shall be a complex Banach space. Replacing $\mathbb{R}$, Z shall be a real Banach space and $\Lambda \subseteq Z$ open the set of parameters, $\lambda_0 \in \Lambda$. Thus we consider $A \in C^1(\Lambda, \mathcal{L}(X;X))$.

Recall: Let $A \in \mathcal{L}(X;X)$ and $\lambda \in \mathbb{C}$. Define the generalized eigenspace to the eigenvalue λ by

$$H_\lambda := \bigcup_{n\in\mathbb{N}} \ker((A - \lambda I)^n)$$

and its algebraic multiplicity $m_a(\lambda) := \dim H_\lambda$. We call λ **algebraically simple** if $m_a(\lambda) = 1$.

We assume that

(A1) μ_0 is an algebraically simple eigenvalue of $A(\lambda_0)$.

(A2) $A(\lambda_0) - \mu_0 I$ is a Fredholm operator of Index 0.

Here, Assumption (A2) is necessary to employ the Implicit Function Theorem. In the case $X = \mathbb{C}^d$ it follows from (A1), which can be easiest seen by considering the Jordan normal form of $A(\lambda_0) - \mu_0 I$.

We will show the following:

<u>*Claim:*</u> There exists open neighborhoods of $U \subseteq \Lambda$ of λ_0, $V \subseteq \mathbb{C}$ of μ_0 and functions $\hat{\mu} \in C^1(U;V)$, $\hat{x} \in C^1(U; X \setminus \{0\})$ such that:

(a) For $\lambda \in U$, $\mu \in V$, μ lies in the spectrum of $A(\lambda)$ if and only if $\mu = \hat{\mu}(\lambda)$.

(b) For $\lambda \in U$, $\hat{\mu}(\lambda)$ is an algebraically simple eigenvalue of $A(\lambda)$.

(c) For $\lambda \in U$, $\ker(A(\lambda) - \hat{\mu}(\lambda)I) = \mathbb{C}\hat{x}(\lambda)$.

Proof: In the following, we abbreviate $B \coloneqq A(\lambda_0) - \mu_0 I$. As 0 is an algebraically simple eigenvalue of B, we have

$$\mathbb{C}x_0 = \ker(B) = \ker(B^2) = \cdots = \ker(B^n)$$

for all $n \in \mathbb{N}$, where $x_0 \in X \setminus \{0\}$.

Step 1: Setup for Implicit Function Theorem. We set $R \coloneqq \operatorname{ran}(B)$.

As B is Fredholm of Index 0, R is closed and of codimension 1. We next show that $X = R \oplus \mathbb{C}x_0$

First we show that $R \cap \mathbb{C}x_0 = \{0\}$. Assume for a contradiction that $x_0 \in R$. Then there exists $x \in X$ with $Bx = x_0$. We have $x \in \ker(B^2) \setminus \ker(B)$, but this set is empty by the assumption on algebraic simplicity, which yields the desired contradiction.

This shows that the map $R \oplus \mathbb{C}x_0 \to X, (r, cx_0) \mapsto (r + cx_0)$ is injective. As R has codimension 1, it is also surjective. Clearly it is bounded, so by the Bounded Inverse theorem it is an isomorphism. In this sense we have $X = R \oplus \mathbb{C}x_0$.

Finally, we formulate the eigenvalue problem as a zero problem by defining

$$F\colon \Lambda \times R \times \mathbb{C} \to X, (\lambda, r, \mu) \mapsto (A(\lambda) - \mu I)[x_0 + r].$$

First we have $F \in C^1(\Lambda \times \mathbb{R} \times \mathbb{C}; X)$ as a composition of C^1 functions and

$$F'(\lambda, r, \mu)[(\tilde{\lambda}, \tilde{r}, \tilde{\mu})] = A'(\lambda)[\tilde{\lambda}][x_0 + r] + (A(\lambda) - \mu I)\tilde{r} - \tilde{\mu}(x_0 + r).$$

Further we have $F(\lambda_0, 0, \mu_0) = 0$ and, if $F(\lambda, r, \mu) = 0$ then μ is an eigenvalue of $A(\lambda)$ and $x_0 + r \neq 0$ is an associated eigenvector. We have

$$F_{(r,\mu)}(\lambda_0, 0, \mu_0)\colon R \times \mathbb{C} \to X, (\tilde{r}, \tilde{\mu}) \mapsto B\tilde{r} - \tilde{\mu}x_0.$$

which is bijective as $B\colon R \to R$ is bijective and $X = R \oplus \mathbb{C}x_0$. So it is invertible by the Bounded Inverse theorem.

Step 2: Applying the Implicit Function Theorem. Thus we have checked all assumptions of the Implicit Function Theorem and we may employ it to obtain open neighborhoods $U_1 \subseteq \Lambda$ of λ_0, $O_1 \subseteq R$ of 0, $V_1 \subseteq \mathbb{C}$ of μ_0, and maps $\hat{r} \in C^1(U_1; O_1)$, $\hat{\mu} \in C^1(U_1; V_1)$ such that for $\lambda \in U_1, r \in O_1$ and $\mu \in V_1$ we have

$$F(\lambda, r, \mu) = 0 \iff r = \hat{r}(\lambda), \mu = \hat{\mu}(\lambda).$$

It is important to note that $\hat{x}(\lambda) \coloneqq x_0 + \hat{r}(\lambda)$ is an eigenvector of $A(\lambda)$ with eigenvalue $\hat{\mu}(\lambda)$ by definition.

Step 3: Discussing the Spectrum. We define the projection operator onto $\mathbb{C}x_0$ with kernel R by $P\colon X \to X, r + cx_0 \mapsto cx_0$. Then,

$$B + P = A(\lambda_0) - \mu_0 I + P \quad \text{ and } B^2 + P = B^2 + P = (A(\lambda_0) - \mu_0 I)^2 + P$$

are invertible in $\mathcal{L}(X; X)$. Therefore, there exist neighborhoods $U_2 \subseteq \Lambda$ of λ_0 and $V_2 \subseteq \mathbb{C}$ of μ_0 such that

$$A(\lambda) - \mu I + P \quad \text{ and } \quad (A(\lambda) - \mu I)^2 + P$$

are invertible for $\lambda \in U_2$ and $\mu \in V_2$.

We begin by making the following observation:

Claim: Let $\lambda \in U_2$ and $\mu \in V_2$. If μ lies in the spectrum of $A(\lambda)$, then μ already must be an algebraically simple eigenvalue of $A(\lambda)$.

To show this, we first observe that P is compact as it is bounded and has finite-dimensional range. Thus

$$A(\lambda) - \mu I = (A(\lambda) - \mu I + P) - P$$

is a Fredholm operator of Index 0. We know that 0 lies in its spectrum, so 0 lies in its point spectrum by the Fredholm property, i.e. μ is an eigenvalue of $A(\lambda)$.
It remains to show that the eigenvalue μ is algebraically simple. From Linear Algebra we know that it suffices to show that $\ker((A(\lambda)-\mu I)^2)$ has dimension 1. Assume for a contradiction that the dimension is at least 2. Then there exists $r \in R\setminus\{0\}$ such that $(A(\lambda)-\mu I)^2 r = 0$. It follows that $((A(\lambda)-\mu I)^2+P)r = 0$, whence $r = 0$. So the claim holds.

Next we show:

Claim: There exist neighborhoods $U_3 \subseteq \Lambda$ of λ_0 and $V_3 \subseteq \mathbb{C}$ of μ_0 such that for $\lambda \in U_3$, if $\mu \in U_3$ is an algebraically simple eigenvalue of $A(\lambda)$, then $\ker(A(\lambda)-\mu I) = \mathbb{C}(x_0 + r)$ where $r \in O$.

Recall that $O \subseteq R$ is some open neighborhood of 0, obtained by the previous application of the IFT. Informally, the claim states that any eigenvector of $A(\lambda)$ must be near $\mathbb{C}x_0$. Thus we can instead show the following claim:

Claim: Let $\lambda_n \in \Lambda, \mu_n \in \mathbb{C}, x_n \in X$ such that (μ_n, x_n) is an eigenpair of $A(\lambda_n)$, $\|x_n\| = 1$ and $\lambda_n \to \lambda_0$, $\mu_n \to \mu_0$ as $n \to \infty$. Then $x_n - Px_n \to 0$ in R as $n \to \infty$.

First, for n sufficiently large $A(\lambda_n) - \mu_n I + P$ is invertible. This allows us to write

$$\begin{aligned} x_n - Px_n &= (A(\lambda_n) - \mu_n I + P)^{-1}(A(\lambda_n) - \mu_n I + P)(I-P)x_n \\ &= (A(\lambda_n) - \mu_n I + P)^{-1}((A(\lambda_n) - \mu_n I + P)(I-P) - (A(\lambda_n) - \mu_n I))x_n \\ &= (A(\lambda_n) - \mu_n I + P)^{-1}(\mu_n P - A(\lambda_n)P)x_n, \end{aligned}$$

so we get

$$\begin{aligned} \|x_n - Px_n\| &\le \left\|(A(\lambda_n) - \mu_n I + P)^{-1}\right\| \|\mu_n P - A(\lambda_n)P\| \|x_n\| \\ &\to \left\|(A(\lambda_0) - \mu_0 I + P)^{-1}\right\| \|\mu_0 P - A(\lambda_0)P\| = 0 \end{aligned}$$

as $n \to \infty$.

Step 4: Putting it all together.

Define $U := U_1 \cap U_2 \cap U_3$ and $V := V_1 \cap V_2 \cap V_3$ and let $\lambda \in U$, $\mu \in V$. If μ lies in the spectrum of $A(\lambda)$, then we have shown that μ is an algebraically simple eigenvalue of $A(\lambda)$, and $\ker(A(\lambda)-\mu I) = \mathbb{C}(x_0 + r)$ where $r \in O$. So $F(\lambda, r, \mu) = 0$, so that by the result of the Implicit Function Theorem $r = \hat{r}(\lambda)$ and $\mu = \hat{\mu}(\lambda)$ must hold. This shows " $\Longrightarrow$ " of part (a).

Conversely, $\hat{\mu}(\lambda)$ is an eigenvalue of $A(\lambda)$ and has eigenvector $\hat{x}(\lambda) = x_0 + \hat{r}(\lambda)$. In particular, $\hat{\mu}(\lambda)$ lies in the spectrum of $A(\lambda)$ (i.e. " $\Longleftarrow$ " of part (a) holds) and again it is an algebraically simple eigenvalue, i.e. parts (b) and (c) hold. □

Remark: If $X = \mathbb{C}^d$ as stated in the problem, we can use the characteristic polynomial $\mathrm{cp}_{A(\lambda)}(\mu) := \det(\mu I - A(\lambda))$ of $A(\lambda)$ for an alternative (and much easier) proof.

For parts (a) and (b) one could look for zeros of $F(\lambda,\mu) = \mathrm{cp}_{A(\lambda)}(\mu)$. Use that $\mathrm{cp}_{A(\lambda_0)}(\mu_0) = 0$, $\mathrm{cp}'_{A(\lambda_0)}(\mu_0) \neq 0$ hold if and only if μ_0 is an algebraically simple eigenvalue of $A(\lambda_0)$.

For part (c) one could for example consider $F(\lambda, x, \mu) = (Ax - \mu x, \langle x - x_0 | x_0 \rangle)$ for $x \in \mathbb{C}^d$, where $x_0 \neq 0$ is an eigenvector of $A(\lambda_0)$ to eigenvalue μ_0.

Problem 12:
Let $\alpha \in (0,1)$, $\emptyset \neq \Omega \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\gamma \in C^{2,\alpha}(\partial\Omega)$. Consider the minimal surface boundary value problem:

$$\begin{cases} (1+u_y^2)u_{xx} + (1+u_x^2)u_{yy} - 2u_x u_y u_{xy} = 0 & \text{in } \Omega, \\ u = \gamma & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Prove that for $\gamma \in C^{2,\alpha}(\partial\Omega)$ sufficiently small there exists a solution $u \in C^{2,\alpha}(\overline{\Omega})$ of (2).

You may use without proof that for all $g \in C^{\alpha}(\overline{\Omega})$ and $h \in C^{2,\alpha}(\partial\Omega)$ the problem

$$\begin{cases} \Delta u = g & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ and that there exists $C > 0$ such that

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \le C\Big(\|g\|_{C^{\alpha}(\overline{\Omega})} + \|h\|_{C^{2,\alpha}(\partial\Omega)}\Big).$$

For a proof, see *[D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 1977]*, Theorem 6.14, the end of chapter 6.3, and Lemma 6.38.

Solution to problem 12:
Let

$$M(u) \coloneqq (1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy}.$$

We intend to apply the Implicit Function Theorem (Theorem 3.5) and thus define $X \coloneqq Y \coloneqq C^{2,\alpha}(\overline{\Omega})$, $Z \coloneqq C^{\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega)$ and consider the operator

$$F\colon X \times Y \to Z, \quad F(\gamma, u) = ((1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy}, u|_{\partial\Omega} - \gamma).$$

Note that X, Y and Z are Banach spaces and F is well-defined with $F(0,0) = (0,0)$. Also, F is Fréchet differentiable (why?) with

$$F'(\gamma, u)[(\phi, w)] = (\diamond, w|_{\partial\Omega} - \phi)$$

where

$$\begin{aligned} \diamond \coloneqq\ & (1 + 2u_y w_y)u_{xx} + (1 + u_y^2)w_{xx} \\ & + (1 + 2u_x w_x)u_{yy} + (1 + u_x^2)w_{xx} \\ & - 2(w_x u_y u_{xy} + u_x w_y u_{xy} + u_x u_y w_{xy}). \end{aligned}$$

In particular, we have $F_u(0,0)[w] = (w_{xx} + w_{yy}, w|_{\partial\Omega}) = (\Delta w, w|_{\partial\Omega})$.

From the hint we know that $F_u(0,0)$ is continuously invertible, so the Implicit Function Theorem can be applied and yields the desired result.