Karlsruhe Institute of Technology

Institute for Analysis
PD Dr. Rainer Mandel
M.Sc. Sebastian Ohrem

Problem Sheet 5

Bifurcation Theory

Winter Semester 2022/23

28.11.2022

Problem 13:
Let $n \in \mathbb{N}$. Consider a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$ and, for $\varepsilon \in \mathbb{R}$, the boundary value problem

$$(1)_\varepsilon \qquad \begin{cases} -\Delta u = u^3 & \text{in } \Omega, \\ u \equiv \varepsilon & \text{on } \partial\Omega. \end{cases}$$

Prove that for $\alpha \in (0,1)$ there exists $\varepsilon_0 > 0$ with the property that problem $(1)_\varepsilon$ admits a classical solution $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Hint: Use the auxiliary result below Problem 12.

Problem 14 (Compact operators I):

Definition. *Let X, Y be Banach spaces. An operator $T \in \mathcal{L}(X;Y)$ is called **compact** if for all bounded sequences (x_n) in X, the image sequence (Tx_n) has a convergent subsequence.*

We denote by $\mathcal{K}(X;Y) \subseteq \mathcal{L}(X;Y)$ the set of compact operators.

Let X, Y, Z be Banach spaces.

(a) Show that if $T \in \mathcal{L}(X;Y)$ has finite-dimensional range, i.e. $\dim(\operatorname{ran} X) < \infty$, then $T \in \mathcal{K}(X;Y)$.

(b) Show that $\mathcal{K}(X;Y) \subseteq \mathcal{L}(X;Y)$ is a linear subspace.

(c) Show that $\mathcal{K}(X;Y) \subseteq \mathcal{L}(X;Y)$ is closed.

(d) Let $S \in \mathcal{L}(X;Y)$ and $T \in \mathcal{L}(Y;Z)$. Show that if S or T are compact, then TS is compact.

(e) Let $T \in \mathcal{K}(X;Y)$ and $x_n \rightharpoonup x$ in X. Prove that $Tx_n \to Tx$.

Problem 15 (Compact operators II):

(a) Show that the embedding $C^2([0,1]) \hookrightarrow C^1([0,1])$ is compact.

(b) Let $k \in C([0,1]^2)$. Show that the following map is compact:

$$T\colon C([0,1]) \to C([0,1]), \quad Tf(x) = \int_0^1 k(x,y) f(y)\,\mathrm{d}y$$

(c) For $y \in \mathbb{R}$ define the translation $\tau_y \in \mathcal{L}(L^1(\mathbb{R}); L^1(\mathbb{R})), \tau_y f(x) = f(x-y)$. Show that any $T \in \mathcal{L}(L^1(\mathbb{R}), L^1(\mathbb{R})) \setminus \{0\}$ which satisfies $\tau_y T = T \tau_y$ for all $y \in \mathbb{R}$ is not compact.

(d) Show that $T\colon \ell^2 \to \ell^2, (x_n) \mapsto (\frac{x_n}{n})$ is compact.

Hint for (a) and (b): Use the Arzelà-Ascoli theorem.