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# Solution to Problem Sheet 5

## Bifurcation Theory

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**Problem 13:**
Let $n \in \mathbb{N}$. Consider a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$ and, for $\varepsilon \in \mathbb{R}$, the boundary value problem

$$(1)_\varepsilon \qquad \begin{cases} -\Delta u = u^3 & \text{in } \Omega, \\ u \equiv \varepsilon & \text{on } \partial\Omega. \end{cases}$$

Prove that for $\alpha \in (0,1)$ there exists $\varepsilon_0 > 0$ with the property that problem $(1)_\varepsilon$ admits a classical solution $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

*Hint:* Use the auxiliary result below Problem 12.

**Solution to problem 13:**
We intend to apply the Implicit Function Theorem. To do this, define the function $F\colon X \times Y \to Z$ with $X = \mathbb{R}$, $Y = C^{2,\alpha}(\overline{\Omega})$, $Z = C^\alpha(\overline{\Omega}) \times C^\alpha(\partial\Omega)$ given by

$$F(\varepsilon, u) = (\Delta u + u^3, u|_{\partial\Omega} - \varepsilon)$$

Then $(\varepsilon, u)$ solve $(1)_\varepsilon$ if and only if $F(\varepsilon, u) = (0,0)$. Further, we have $F(0,0) = (0,0)$.

For the estimates proving continuous differentiability, we need an algebraic property of the Hölder norms.

**Lemma.** *For $a, b \in C^{0,\alpha}(\overline{\Omega})$, we have that $ab \in C^{0,\alpha}(\overline{\Omega})$ with*

$$\|ab\|_{C^{0,\alpha}(\overline{\Omega})} \le 3\|a\|_{C^{0,\alpha}(\overline{\Omega})}\|b\|_{C^{0,\alpha}(\overline{\Omega})}. \qquad (\diamondsuit)$$

*Proof of Lemma:* Let $a, b \in C^{0,\alpha}(\overline{\Omega})$. First, we note that $ab$ is a continuous function. We now estimate the norms, recalling that

$$\|a\|_{C^{0,\alpha}(\overline{\Omega})} = \|a\|_\infty + [a]_{0,\alpha} = \sup_{x \in \overline{\Omega}} |a(x)| + \sup_{x \neq y \in \overline{\Omega}} \frac{|a(x) - a(y)|}{|x-y|^\alpha}.$$

We have $\|ab\|_\infty \le \|a\|_\infty \|b\|_\infty \le \|a\|_{C^{0,\alpha}(\overline{\Omega})}\|b\|_{C^{0,\alpha}(\overline{\Omega})}$ and

$$\begin{aligned} [ab]_{0,\alpha} &= \sup_{x \neq y \in \overline{\Omega}} \frac{|a(x)b(x) - a(y)b(y)|}{|x-y|^\alpha} \le \sup_{x \neq y \in \overline{\Omega}} \frac{|a(x)||b(x) - b(y)| + |a(x) - a(y)||b(y)|}{|x-y|^\alpha} \\ &\le \|a\|_\infty \sup_{x \neq y \in \overline{\Omega}} \frac{|b(x) - b(y)|}{|x-y|^\alpha} + \|b\|_\infty \sup_{x \neq y \in \overline{\Omega}} \frac{|a(x) - a(y)|}{|x-y|^\alpha} \\ &= \|a\|_\infty [b]_{0,\alpha} + \|b\|_\infty [a]_{0,\alpha} \le 2\|a\|_{C^{0,\alpha}(\overline{\Omega})}\|b\|_{C^{0,\alpha}(\overline{\Omega})}, \end{aligned}$$

and summing up both estimates, the lemma is proved. □

**Step 1: Fréchet differentiability of $F$.** We claim that $F$ is continuously Fréchet differentiable with

$$(1) \qquad F'(\varepsilon, u)[(\delta, h)] = (\Delta h + 3u^2 h, h|_{\partial\Omega} - \delta)$$

To show this, we estimate

$$F(\varepsilon + \delta, u + h) - F(\varepsilon, u) = (\Delta(u+h) + (u+h)^3 - \Delta u - u^3, (u+h)|_{\partial\Omega} - (\varepsilon + \delta) - u|_{\partial\Omega} + \varepsilon)$$

$$= (\Delta h + 3u^2 h, h|_{\partial\Omega} - \delta) + (3uh^2 + h^3, 0).$$

Using that there exists a constant $C > 0$ such that $\|w\|_{C^{0,\alpha}(\overline{\Omega})} \leq C\|w\|_{C^{2,\alpha}(\overline{\Omega})}$, we estimate

$$\begin{aligned}
\left\|(3uh^2 + h^3, 0)\right\|_Z &= \left\|3uh^2 + h^3\right\|_{C^{0,\alpha}(\overline{\Omega})} \\
&\leq 9\Big(3\|u\|_{C^{0,\alpha}(\overline{\Omega})}\|h\|^2_{C^{0,\alpha}(\overline{\Omega})} + \|h\|^3_{C^{0,\alpha}(\overline{\Omega})}\Big) \\
&\leq 9C^3(3\|u\|_{C^{2,\alpha}(\overline{\Omega})} + \|h\|_{C^{2,\alpha}(\overline{\Omega})})\|h\|^2_{C^{2,\alpha}(\overline{\Omega})} = \mathcal{O}\Big(\|h\|_Y^2\Big) = \mathrm{o}\big(\|(\delta, h)\|_{X\times Y}\big)
\end{aligned}$$

as $(\delta, h) \to 0$. So $F$ is Fréchet differentiable and the derivative is given by (1).

It remains to show that the derivative is continuous. To do this, let $\varepsilon, \tilde{\varepsilon}, \delta \in X$ and $u, \tilde{u}, h \in Y$. We calculate

$$\begin{aligned}
&\|F'(\varepsilon + \tilde{\varepsilon}, u + \tilde{u})[(\delta, h)] - F'(\varepsilon, u)[(\delta, h)]\|_Z \\
&\quad = \left\|(3(2u\tilde{u} + \tilde{u}^2)h, 0)\right\|_Z \\
&\quad \leq 27\|\tilde{u}\|_{C^{0,\alpha}(\overline{\Omega})}\Big(2\|u\|_{C^{0,\alpha}(\overline{\Omega})} + \|\tilde{u}\|_{C^{0,\alpha}(\overline{\Omega})}\Big)\|h\|_{C^{0,\alpha}(\overline{\Omega})} \\
&\quad \leq 27C^3\|\tilde{u}\|_{C^{2,\alpha}(\overline{\Omega})}\Big(2\|u\|_{C^{2,\alpha}(\overline{\Omega})} + \|\tilde{u}\|_{C^{2,\alpha}(\overline{\Omega})}\Big)\|h\|_{C^{2,\alpha}(\overline{\Omega})} = \mathrm{o}(\|(\delta, h)\|_Z)
\end{aligned}$$

as $(\tilde{\varepsilon}, \tilde{u}) \to (0, 0)$ in $Z$, so $F'$ is continuous in $(\varepsilon, u)$. As $(\varepsilon, u)$ are chosen arbitrarily, $F'$ is continuous.

**Step 2: Applying the IFT.** Recall that $F_u(0, 0)$ is given by

$$F_u(0, 0)[h] = (\Delta h, h|_{\partial\Omega}).$$

The auxiliary result below Problem 12 tells us that $F_u(0, 0)\colon C^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega}) \times C^{0,\alpha}(\partial\Omega)$ is continuously invertible.

We have verified all assumptions of the Implicit Function Theorem, so that we may now apply it to obtain $\varepsilon_0 > 0$ and a function $\hat{u} \in C^1((-\varepsilon_0, \varepsilon_0); Y)$ such that $F(\varepsilon, \hat{u}(\varepsilon)) = 0$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Setting $u_\varepsilon := \hat{u}(\varepsilon)$ completes the proof.

**Problem 14 (Compact operators I):**

**Definition.** *Let $X, Y$ be Banach spaces. An operator $T \in \mathcal{L}(X; Y)$ is called **compact** if for all bounded sequences $(x_n)$ in $X$, the image sequence $(Tx_n)$ has a convergent subsequence.*

We denote by $\mathcal{K}(X; Y) \subseteq \mathcal{L}(X; Y)$ the set of compact operators.

Let $X, Y, Z$ be Banach spaces.

(a) Show that if $T \in \mathcal{L}(X; Y)$ has finite-dimensional range, i.e. $\dim(\operatorname{ran} X) < \infty$, then $T \in \mathcal{K}(X; Y)$.

(b) Show that $\mathcal{K}(X; Y) \subseteq \mathcal{L}(X; Y)$ is a linear subspace.

(c) Show that $\mathcal{K}(X; Y) \subseteq \mathcal{L}(X; Y)$ is closed.

(d) Let $S \in \mathcal{L}(X; Y)$ and $T \in \mathcal{L}(Y; Z)$. Show that if $S$ or $T$ are compact, then $TS$ is compact.

(e) Let $T \in \mathcal{K}(X; Y)$ and $x_n \rightharpoonup x$ in $X$. Prove that $Tx_n \to Tx$.

**Solution to problem 14:**

(a) <u>*Proof:*</u> Let $(x_n)$ in $X$ be a bounded sequence such that $\|x_n\| \leq R$ for $n \in \mathbb{N}$. Then $\|Tx_n\| \leq \|T\|\|x_n\| \leq \|T\|R$, so that $Tx_n \in TX$ is bounded. As $TX$ has finite dimension, by the Heine-Borel theorem there exists a convergent subsequence of $(Tx_n)$. ☐

(b) <u>*Proof:*</u> Let $T, S \in \mathcal{K}(X; Y)$, $\mu \in \mathbb{R}$, and $(x_n)$ be an arbitrary bounded sequence in $X$. Then there exists a subsequence $(x_{n_k})$ such that $Tx_{n_k}$ and $Sx_{n_k}$ converge. Then also $(T + \mu s)x_{n_k}$ converges. So $T + \mu S \in \mathcal{K}(X; Y)$. ☐

(c) *Proof:* Let $(x_k)$ be bounded in $X$ with $\|x_k\| \leq R$. Using compactness of the $T_n$ and a diagonal sequence argument we find a subsequence $(x_{k_l})$ such that $T_n x_{k_l}$ converges for all $n \in \mathbb{N}$. We then estimate

$$\left\|Tx_{k_l} - Tx_{k_j}\right\| \leq \|Tx_{k_l} - T_n x_{k_l}\| + \left\|T_n x_{k_l} - T_n x_{k_j}\right\| + \left\|T_n x_{k_j} - Tx_{k_j}\right\| \leq 2\|T - T_n\|R + \left\|T_n x_{k_l} - T_n x_{k_j}\right\|.$$

Fix $\varepsilon > 0$ and choose $n$ so that $\|T - T_n\| \leq \frac{\varepsilon}{4R}$. As $T_n x_{k_l}$ converges, there exists $N \in \mathbb{N}$ so that $\left\|T_n x_{k_l} - T_n x_{k_j}\right\| \leq \frac{\varepsilon}{2}$ for $l, j \geq N$. Thus $\left\|Tx_{k_l} - Tx_{k_j}\right\| \leq \varepsilon$ for $l, j \geq N$, showing that $(Tx_{k_l})$ is a Cauchy sequence. By completeness of $\mathcal{L}(X; Y)$ it converges. □

(d) *Proof:* Assume first that $S$ is compact and $(x_n)$ is bounded in $X$. Then there exists a convergent subsequence $Sx_{n_k} \to y$. As $T$ is bounded, we have $TSx_{n_k} \to Ty$, showing that also $TS$ is compact.

Now, if $T$ is compact and $(x_n)$ is bounded in $X$, then $(Sx_n)$ is bounded in $Y$ by continuity of $S$, so that by compactness of $T$ there exists a convergent subsequence $(TSx_{n_k})$. So again $TS$ is compact. □

(e) *Proof:* First, we have $Tx_n \rightharpoonup Tx$ as $n \to \infty$. Let $\varphi \in Y'$. Then we have

$$\varphi(Tx_n) = \langle Tx_n, \varphi\rangle = \langle x_n, T'\varphi\rangle \to \langle x, T'\varphi\rangle = \langle Tx, \varphi\rangle = \varphi(Tx)$$

as $n \to \infty$, so that $Tx_n$ converges weakly to $Ty$. Now let $(x_{n_k})$ be an arbitrary subsequence of $(x_n)$. By compactness, there exists a convergent subsequence $Tx_{n_{k_l}} \to y$. But then also $Tx_{n_{k_l}} \rightharpoonup y$, so that $y = Tx$, i.e. $Tx_{n_{k_l}} \to Tx$. It follows that $Tx_n \to Tx$. □

**Problem 15 (Compact operators II):**

(a) Show that the embedding $C^2([0,1]) \hookrightarrow C^1([0,1])$ is compact.

(b) Let $k \in C([0,1]^2)$. Show that the following map is compact:

$$T\colon C([0,1]) \to C([0,1]), \quad Tf(x) = \int_0^1 k(x,y)f(y)\,\mathrm{d}y$$

(c) For $y \in \mathbb{R}$ define the translation $\tau_y \in \mathcal{L}(L^1(\mathbb{R}); L^1(\mathbb{R})), \tau_y f(x) = f(x - y)$. Show that any $T \in \mathcal{L}(L^1(\mathbb{R}), L^1(\mathbb{R})) \setminus \{0\}$ which satisfies $\tau_y T = T\tau_y$ for all $y \in \mathbb{R}$ is not compact.

(d) Show that $T\colon \ell^2 \to \ell^2, (x_n) \mapsto (\frac{x_n}{n})$ is compact.

*Hint for (a) and (b):* Use the Arzelà-Ascoli theorem.

**Solution to problem 15:**

(a) *Proof:* Let $f_n \in C^2([0,1])$ be a bounded sequence, i.e. $\|f_n\|_\infty, \|f_n'\|_\infty, \|f_n''\|_\infty \leq R$. Then $f_n, f_n'$ are Lipschitz continuous with Lipschitz constant $R$. In particular, they are uniformly bounded and uniformly equicontinuous. By the Arzelà-Ascoli theorem there exists a convergent subsequence $f_{n_k} \to f, f_{n_k}' \to g$ in $C([0,1])$ as $n \to \infty$. Since

$$f(b) - f(a) = \lim_{n\to\infty} f_n(b) - f_n(a) = \lim_{n\to\infty} \int_a^b f_n'(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n'(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x$$

We have $f \in C^1([0,1])$ and $f' = g$. So we have shown $f_{n_k} \to f$ in $C^1([0,1])$ as $n \to \infty$. □

(b) *Proof:* Let $f_n \in C([0,1])$ be bounded, $\|f_n\|_\infty \leq R$. Let $\varepsilon > 0$. As $[0,1]^2$ is compact, $k$ is uniformly continuous, so there exists $\delta > 0$ such that for $x_1, x_2, y_1, y_2 in [0,1]$ with $|x_1 - x_2|, |y_1 - y_2| < \delta$ we have $|k(x_1, y_1) - k(x_2, y_2)| < \varepsilon$. For $x_1, x_2 \in [0,1]$ with $|x_1 - x_2| < \delta$ we estimate

$$|Tf_n(x_1) - Tf_n(x_2)| \leq \int_0^1 |k(x_1,y)f_n(y) - k(x_2,y)f_n(y)|\,\mathrm{d}y \leq \varepsilon R,$$

showing that the $Tf_n$ are uniformly equicontinuous. Since $\|Tf_n\| \leq \|k\|_\infty R$ they are also uniformly bounded. Hence the Arzelà-Ascoli theorem tells us that $(Tf_n)$ has a uniformly convergent subsequence. □

(c) <u>*Proof:*</u> Fix $f \in L^1(\mathbb{R}) \setminus \ker T$ and let $g := Tf \in L^1(\mathbb{R}) \setminus \{0\}$. As

$$\|g\|_1 = \lim_{R\to\infty} \int_{-R}^{R} |g| \,\mathrm{d}x$$

there exists $R > 0$ such that

$$\int_{-R}^{R} |g| \,\mathrm{d}x \geq \frac{3}{4}\|g\|_1.$$

We now choose $f_n := f(\,\cdot\, - 2nR)$ for $n \in \mathbb{N}$. Then $f_n$ is bounded in $L^1(\mathbb{R})$ with $\|f_n\|_1 = \|f\|$ for $n \in \mathbb{N}$. However, $Tf_n$ does not admit convergent subsequences. In fact, for $m, n \in \mathbb{N}$ with $m \neq n$ we estimate

$$\begin{aligned}
\|Tf_n - Tf_m\|_1 &= \|T\tau_{2nR}f - T\tau_{2mR}f\|_1 \\
&= \|\tau_{2nR}Tf - \tau_{2mR}Tf\|_1 \\
&= \|g(\,\cdot\, - 2nR) - g(\,\cdot\, - 2mR)\|_1 \\
&= \int_{(2n-1)R}^{2n+1R} |g(x-2nR) - g(x-2mR)| \,\mathrm{d}x + \int_{(2m-1)R}^{2m+1R} |g(x-2nR) - g(x-2mR)| \,\mathrm{d}x \\
&\geq \int_{-R}^{R} |g(x)| - |g(x+2(n-m)R)| \,\mathrm{d}x + \int_{-R}^{R} |g(x)| - |g(x+2(m-n)R)| \,\mathrm{d}x \\
&\geq 2\int_{-R}^{R} |g(x)| \,\mathrm{d}x - 2\int_{\mathbb{R}\setminus(-R,R)} |g(x)| \,\mathrm{d}x \geq \|g\|_1 > 0.
\end{aligned}$$

□

(d) <u>*Proof:*</u> For $x \in \ell^2$ define the operator $T_k \colon \ell^2 \to \ell^2$ by

$$(T_k x)_n = \begin{cases} \frac{x_n}{n}, & n < k, \\ 0, & n \geq k \end{cases}$$

Then $T_k \in \mathcal{L}(\ell^2; \ell^2)$ is compact (as it has finite-dimensional range) and satisfies $\|T - T_k\| = \frac{1}{k}$ for all $k$, so that $T = \lim_{k\to\infty} T_k$ is also compact. □