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Problem Sheet 6

Bifurcation Theory

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Problem 16:
Determine all bifurcation points $(0, \lambda_0) \in \mathbb{R}^2 \times \mathbb{R}$ of the nonlinear system

$$\begin{cases} \sin(x_1 + \lambda x_2) = x_1, \\ \cos(\lambda x_1 + x_2) = 1 + x_1. \end{cases} \tag{1}$$

Problem 17:
Let $A \in \mathbb{R}^{n \times n}$ be symmetric, and $J \in \mathbb{R}^{n \times n}$. For $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we study the equation

$$Ax = \lambda x + |x|^2 Jx. \tag{2}$$

Discuss the existence of nontrivial solutions in a neighborhood of $(0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ if ...

(a) λ_0 is not an eigenvalue of A,

(b) λ_0 is a simple eigenvalue of A.

Problem 18:
Let $g \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}; \mathbb{R}), (x, z, \lambda) \mapsto g(x, z, \lambda)$ be 2π-periodic in x with

$$g(x, 0, \lambda) = 0, \quad g_z(x, 0, \lambda) = 0, \quad g_{z\lambda}(x, 0, \lambda) = 0 \quad \text{for all } x \in \mathbb{R}, \lambda \in \mathbb{R}.$$

In order to find nontrivial 2π-periodic solutions $u \in C^2(\mathbb{R})$ of the ODE

$$-u'' = \lambda u + g(\,\cdot\,, u, \lambda) \qquad \text{on } \mathbb{R} \tag{3}$$

in a neighborhood of $(u_0, \lambda_0) = (0, 0)$, proceed as follows:

(a) Let $F : C^2_{\text{per}}(\mathbb{R}) \times \mathbb{R} \to C_{\text{per}}(\mathbb{R}), F(u, \lambda) := u'' + \lambda u + g(\,\cdot\,, u, \lambda)$ where

$$C^k_{\text{per}}(\mathbb{R}) := \left\{ u \in C^k(\mathbb{R}) \colon u(x) = u(x + 2\pi) \text{ for all } x \in \mathbb{R} \right\} \qquad \text{for } k \in \mathbb{N}_0.$$

Show that F is twice continuously Fréchet differentiable and calculate F' and F''.

(b) Show that $\ker(F_u(0, 0)) = \text{span}\{\mathbb{1}\}$; $\text{ran}(F_u(0, 0)) = \left\{ z \in C_{\text{per}}(\mathbb{R}) \colon \int_0^{2\pi} z(t)\, \mathrm{d}t = 0 \right\}$.

(c) Prove that there exist $\delta > 0$ and a continuous branch $(-\delta, \delta) \to C^2_{\text{per}}(\mathbb{R}) \times \mathbb{R},\ s \mapsto (\hat{u}(s), \hat{\lambda}(s))$ with the property that

$$\left\{ (\hat{u}(s), \hat{\lambda}(s)) \colon 0 < |s| < \delta \right\}$$

collects all nontrivial 2π-periodic solutions of problem (3) in a neighborhood of $(0, 0)$.