

Solution to Problem Sheet 6

Bifurcation Theory

Winter Semester 2022/23

12.12.2022

Problem 16:

Determine all bifurcation points $(0, \lambda_0) \in \mathbb{R}^2 \times \mathbb{R}$ of the nonlinear system

$$(1) \quad \begin{cases} \sin(x_1 + \lambda x_2) = x_1, \\ \cos(\lambda x_1 + x_2) = 1 + x_1. \end{cases}$$

Solution to problem 16:

Claim: $(0, \lambda_0)$ is a bifurcation point of problem (1) if and only if $\lambda_0 = 0$.

Proof: Solving problem (1) is equivalent to finding zeros of the function

$$F : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2, \quad F(x_1, x_2, \lambda) := \begin{pmatrix} \sin(x_1 + \lambda x_2) - x_1 \\ \cos(\lambda x_1 + x_2) - 1 - x_1 \end{pmatrix}.$$

Then $F(0, 0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$, and F is twice continuously differentiable with

$$F_x(x_1, x_2, \lambda) = \begin{pmatrix} \cos(x_1 + \lambda x_2) - 1 & \lambda \cos(x_1 + \lambda x_2) \\ -\lambda \sin(\lambda x_1 + x_2) - 1 & -\sin(\lambda x_1 + x_2) \end{pmatrix},$$

$$F_{x\lambda}(x_1, x_2, \lambda) = \begin{pmatrix} -x_2 \sin(x_1 + \lambda x_2) & \cos(x_1 + \lambda x_2) - \lambda x_2 \sin(x_1 + \lambda x_2) \\ -\sin(\lambda x_1 + x_2) - \lambda x_1 \cos(\lambda x_1 + x_2) & -x_1 \cos(\lambda x_1 + x_2) \end{pmatrix}$$

for $x_1, x_2, \lambda \in \mathbb{R}$, and in particular, for $\lambda_0 \in \mathbb{R}$,

$$F_x(0, 0, \lambda_0) = \begin{pmatrix} 0 & \lambda_0 \\ -1 & 0 \end{pmatrix}, \quad F_{x\lambda}(0, 0, \lambda_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If $\lambda_0 \neq 0$, we have that $\det F_x(0, 0, \lambda_0) = \lambda_0 \neq 0$; hence $F_x(0, 0, \lambda_0)$ (interpreted as a linear mapping from \mathbb{R}^2 to \mathbb{R}^2) is a homeomorphism. Proposition 3.9 then states that $(0, \lambda_0)$ cannot be a bifurcation point for problem (1).

Now let $\lambda_0 = 0$ and apply the Crandall-Rabinowitz Bifurcation Theorem: We have

$$\ker F_x(0, 0, 0) = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad \text{ran } F_x(0, 0, 0) = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

and thus, the simplicity condition (S) holds. Moreover, with $\varphi := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have

$$F_{x\lambda}(0, 0, 0)[\varphi] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \text{ran } F_x(0, 0, 0),$$

which shows that the transversality condition (T) is also satisfied. Theorem 4.3 now states that $(0, 0, 0)$ is a bifurcation point. \square

Problem 17:

Let $A \in \mathbb{R}^{n \times n}$ be symmetric, and $J \in \mathbb{R}^{n \times n}$. For $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we study the equation

$$(2) \quad Ax = \lambda x + |x|^2 Jx.$$

Discuss the existence of nontrivial solutions in a neighborhood of $(0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ if ...

- (a) λ_0 is not an eigenvalue of A ,
- (b) λ_0 is a simple eigenvalue of A .

Solution to problem 17:

Solving problem (2) is equivalent to finding zeros of the function

$$F : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n, \quad F(x, \lambda) := Ax - \lambda x - |x|^2 Jx.$$

We note that $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$, and that F is twice continuously differentiable with

$$F_x(x, \lambda)[h] = Ah - \lambda h - |x|^2 Jh - 2\langle x|h \rangle Jx, \quad F_{x\lambda}(x, \lambda)[h] = -h$$

for $x, h, \lambda \in \mathbb{R}$; moreover, for $\lambda_0 \in \mathbb{R}$,

$$F_x(0, \lambda)[h] = (A - \lambda I)h, \quad F_{x\lambda}(0, \lambda)[h] = -h.$$

- (a) Let λ_0 be not an eigenvalue of A . Claim: There are no nontrivial solutions of problem (2) in a neighborhood of $(0, \lambda_0)$.

Proof: This is a consequence of Proposition 3.9, as $F_x(0, \lambda_0) = A - \lambda_0 I : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is invertible. \square

- (b) Let λ_0 be a simple eigenvalue of A . Claim: Nontrivial solutions of problem (2) form a continuous branch bifurcating from $(0, \lambda_0)$.

Proof: We check the assumptions of the Crandall-Rabinowitz Bifurcation Theorem. Since A is assumed symmetric, there exist eigenpairs $(\psi_j, \mu_j) \in \mathbb{R}^n \times \mathbb{R}$ with $A\psi_j = \mu_j\psi_j$ and $\{\psi_1, \dots, \psi_n\}$ being a complete orthonormal subset of \mathbb{R}^n .

Without loss of generality, $\mu_1 = \lambda_0$ and by assumption (simple eigenvalue), we have $\mu_j \neq \lambda_0$ for $j > 1$. Thus,

$$\begin{aligned} \ker F_x(0, \lambda_0) &= \ker(A - \lambda_0 I) = \text{span}\{\psi_1\}, \\ \text{ran } F_x(0, \lambda_0) &= \text{ran}(A - \lambda_0 I) = \text{span}\{\psi_2, \dots, \psi_n\} \end{aligned}$$

and we infer $\dim \ker F_x(0, \lambda_0) = \text{codim } \text{ran } F_x(0, \lambda_0) = 1$, so (S) holds. Moreover,

$$F_{x\lambda}(0, \lambda_0)[\psi_1] = -\psi_1 \notin \text{ran } F_x(0, \lambda_0),$$

and (T) is satisfied. Theorem 4.3 ensures the existence of a unique continuous branch of nontrivial solutions of problem (2) bifurcating from $(0, \lambda_0)$. \square

Problem 18:

Let $g \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, $(x, z, \lambda) \mapsto g(x, z, \lambda)$ be 2π -periodic in x with

$$g(x, 0, \lambda) = 0, \quad g_z(x, 0, \lambda) = 0, \quad g_{z\lambda}(x, 0, \lambda) = 0 \quad \text{for all } x \in \mathbb{R}, \lambda \in \mathbb{R}.$$

In order to find nontrivial 2π -periodic solutions $u \in C^2(\mathbb{R})$ of the ODE

$$(3) \quad -u'' = \lambda u + g(\cdot, u, \lambda) \quad \text{on } \mathbb{R}$$

in a neighborhood of $(u_0, \lambda_0) = (0, 0)$, proceed as follows:

(a) Let $F : C_{\text{per}}^2(\mathbb{R}) \times \mathbb{R} \rightarrow C_{\text{per}}(\mathbb{R})$, $F(u, \lambda) := u'' + \lambda u + g(\cdot, u, \lambda)$ where

$$C_{\text{per}}^k(\mathbb{R}) := \{u \in C^k(\mathbb{R}) : u(x) = u(x + 2\pi) \text{ for all } x \in \mathbb{R}\} \quad \text{for } k \in \mathbb{N}_0.$$

Show that F is twice continuously Fréchet differentiable and calculate F' and F'' .

(b) Show that $\ker(F_u(0, 0)) = \text{span}\{\mathbf{1}\}$; $\text{ran}(F_u(0, 0)) = \left\{z \in C_{\text{per}}(\mathbb{R}) : \int_0^{2\pi} z(t) dt = 0\right\}$.

(c) Prove that there exist $\delta > 0$ and a continuous branch $(-\delta, \delta) \rightarrow C_{\text{per}}^2(\mathbb{R}) \times \mathbb{R}$, $s \mapsto (\hat{u}(s), \hat{\lambda}(s))$ with the property that

$$\left\{(\hat{u}(s), \hat{\lambda}(s)) : 0 < |s| < \delta\right\}$$

collects all nontrivial 2π -periodic solutions of problem (3) in a neighborhood of $(0, 0)$.

Solution to problem 18:

(a) First let $k \in \mathbb{N}_0$. Recall that $C_b^k(\mathbb{R})$ consists of functions $f \in C^k(\mathbb{R})$ such that

$$\|f\|_{C_b^k(\mathbb{R})} = \sup\{|f(x)|, |f'(x)|, \dots, |f^{(k)}(x)| : x \in \mathbb{R}\} < \infty.$$

Since $\tau : C_b^k(\mathbb{R}) \rightarrow C_b^k(\mathbb{R})$, $f \mapsto f(\cdot - 2\pi)$ defines an isometric isomorphism, we see that $C_{\text{per}}^k(\mathbb{R}) = \ker(I - \tau)$ is a closed subspace of $C_b^k(\mathbb{R})$, and thus is a Banach space. We denote $C_{\text{per}}(\mathbb{R}) := C_{\text{per}}^0(\mathbb{R})$.

Claim: F is twice continuously differentiable and

$$\begin{aligned} F'(u, \lambda)[(h_u, h_\lambda)] &= h_u'' + h_\lambda u + \lambda h_u + g_z(\cdot, u, \lambda)h_u + g_\lambda(\cdot, u, \lambda)h_\lambda, \\ F''(u, \lambda)[(h_u, h_\lambda), (\eta_u, \eta_\lambda)] &= h_\lambda \eta_u + \eta_\lambda h_u + h_u \eta_u g_{zz}(\cdot, u, \lambda) + (\eta_\lambda h_u + h_\lambda \eta_u)g_{z\lambda}(\cdot, u, \lambda) + h_\lambda \eta_\lambda g_{\lambda\lambda}(\cdot, u, \lambda). \end{aligned}$$

Proof: We write $F = G + H$ with

$$\begin{aligned} G : C_{\text{per}}^2(\mathbb{R}) \times \mathbb{R} &\rightarrow C_{\text{per}}(\mathbb{R}), G(u, \lambda) = g(\cdot, u, \lambda), \\ H : C_{\text{per}}^2(\mathbb{R}) \times \mathbb{R} &\rightarrow C_{\text{per}}(\mathbb{R}), H(u, \lambda) = u'' + \lambda u. \end{aligned}$$

Using the product rule we see that H is infinitely differentiable with

$$\begin{aligned} H'(u, \lambda)[(h_u, h_\lambda)] &= h_u'' + h_\lambda u + \lambda h_u, \\ H''(u, \lambda)[(h_u, h_\lambda), (\eta_u, \eta_\lambda)] &= h_\lambda \eta_u + \eta_\lambda h_u, \\ H''' &= 0, \dots \end{aligned}$$

In particular, H is twice continuously differentiable.

Next, we show that G is twice continuously differentiable and that

$$\begin{aligned} G'(u, \lambda)[(h_u, h_\lambda)] &= h_u g_z(\cdot, u, \lambda) + h_\lambda g_\lambda(\cdot, u, \lambda), \\ G''(u, \lambda)[(h_u, h_\lambda), (\eta_u, \eta_\lambda)] &= h_u \eta_u g_{zz}(\cdot, u, \lambda) + (\eta_\lambda h_u + h_\lambda \eta_u)g_{z\lambda}(\cdot, u, \lambda) + h_\lambda \eta_\lambda g_{\lambda\lambda}(\cdot, u, \lambda). \end{aligned}$$

To do this, we proceed similar to Problem 7. Let $(u, \lambda), (h_u, h_\lambda), (\eta_u, \eta_\lambda), (\rho_u, \rho_\lambda) \in C_{\text{per}}^2(\mathbb{R}) \times \mathbb{R}$. We choose

$$K := [0, 2\pi] \times [\inf u - 1, \sup u + 1] \times [\lambda - 1, \lambda + 1] \subseteq \mathbb{R}^3,$$

which is compact, so that g, g_z, g_λ, \dots are uniformly continuous on K . Thus there exists an increasing function $\varphi: (0, \infty) \rightarrow [0, \infty]$ with $\lim_{r \rightarrow 0} \varphi(r) = 0$ so that $|g(x, u, \lambda) - g(\tilde{x}, \tilde{u}, \tilde{\lambda})| \leq \varphi(|x - \tilde{x}| + |z - \tilde{z}| + |\lambda - \tilde{\lambda}|)$ for $(x, z, \lambda), (\tilde{x}, \tilde{z}, \tilde{\lambda}) \in K$; and the same estimate holds for derivatives of g .

We then calculate for $\|h_u\|_\infty, |h_\lambda| \leq 1$:

$$\begin{aligned} & \|G(u + h_u, \lambda + h_\lambda) - G(u, \lambda) - h_u g_z(\cdot, u, \lambda) - h_\lambda g_\lambda(\cdot, u, \lambda)\|_\infty \\ &= \left\| \int_0^1 h_u [g_z(\cdot, u + th_u, \lambda + th_\lambda) - g_z(\cdot, u, \lambda)] + h_\lambda [g_\lambda(\cdot, u + th_u, \lambda + th_\lambda) - g_\lambda(\cdot, u, \lambda)] dt \right\|_\infty \\ &\leq \int_0^1 \|h_u\|_\infty \varphi(|t| \|h_u\| + |t| |h_\lambda|) + |h_\lambda| \varphi(|t| \|h_u\| + |t| |h_\lambda|) dt \\ &\leq (\|h_u\|_\infty + |h_\lambda|) \varphi(\|h_u\|_\infty + |h_\lambda|) = o(\|h_u\|_\infty + |h_\lambda|) \end{aligned}$$

as $(h_u, h_\lambda) \rightarrow 0$. So G is Fréchet-differentiable and the derivative is as stated.

Next we confirm the formula for the second derivative. We calculate for $\|\eta_u\|_\infty, |\eta_\lambda| \leq 1$:

$$\begin{aligned} I &:= G'(u + \eta_u, \lambda + \eta_\lambda)[(h_u, h_\lambda)] - G'(u, \lambda)[(h_u, h_\lambda)] \\ &\quad - h_u \eta_u g_{zz}(\cdot, u, \lambda) - (\eta_\lambda h_u + h_\lambda \eta_u) g_{z\lambda}(\cdot, u, \lambda) - h_\lambda \eta_\lambda g_{\lambda\lambda}(\cdot, u, \lambda) \\ &= \int_0^1 h_u (\eta_u [g_{zz}(\cdot, u + t\eta_u, \lambda + t\eta_\lambda) - g_{zz}(\cdot, u, \lambda)] + \eta_\lambda [g_{z\lambda}(\cdot, u + t\eta_u, \lambda + t\eta_\lambda) - g_{z\lambda}(\cdot, u, \lambda)]) dt \\ &\quad + \int_0^1 h_\lambda (\eta_u [g_{\lambda z}(\cdot, u + t\eta_u, \lambda + t\eta_\lambda) - g_{\lambda z}(\cdot, u, \lambda)] + \eta_\lambda [g_{\lambda\lambda}(\cdot, u + t\eta_u, \lambda + t\eta_\lambda) - g_{\lambda\lambda}(\cdot, u, \lambda)]) dt \end{aligned}$$

and thus get

$$\|I\|_\infty \leq (\|h_u\|_\infty + |h_\lambda|)(\|\eta_u\|_\infty + |\eta_\lambda|) \varphi(\|\eta_u\|_\infty + |\eta_\lambda|) = (\|h_u\|_\infty + |h_\lambda|) \cdot o(\|\eta_u\|_\infty + |\eta_\lambda|)$$

as $(\eta_u, \eta_\lambda) \rightarrow 0$. So G is twice Fréchet-differentiable and the derivative is as stated. For continuity of the second derivative, we calculate for $\|\rho_u\|_\infty, |\rho_\lambda| \leq 1$:

$$\begin{aligned} & \|G''(u + \rho_u, \lambda + \rho_\lambda)[(h_u, h_\lambda), (\eta_u, \eta_\lambda)] - G''(u, \lambda)[(h_u, h_\lambda), (\eta_u, \eta_\lambda)]\| \\ &\leq (\|h_u\|_\infty + |h_\lambda|)(\|\eta_u\|_\infty + |\eta_\lambda|) \varphi(\|\rho_u\|_\infty + |\rho_\lambda|) \\ &= (\|h_u\|_\infty + |h_\lambda|)(\|\eta_u\|_\infty + |\eta_\lambda|) \cdot o(1) \end{aligned}$$

as $(\rho_u, \rho_\lambda) \rightarrow 0$. This shows that G is twice continuously Fréchet-differentiable.

The claim now follows from $F = G + H$ and linearity of the derivative operator. \square

(b) Claim: $\ker(F_u(0, 0)) = \text{span}\{\mathbf{1}\}$.

Proof: From part (a) we know that

$$F_u(0, 0)[h] = F'(0, 0)[(h, 0)] = h''$$

for $h \in C_{\text{per}}^2(\mathbb{R})$. For such h , the following are equivalent:

$$\begin{aligned} h \in \ker F_u(0, 0) &\iff h'' = 0 \iff \exists \alpha, \beta \in \mathbb{R} : h(x) = \alpha x + \beta \text{ for } x \in \mathbb{R} \\ &\iff \exists \beta \in \mathbb{R} : h(x) = \beta \text{ for } x \in \mathbb{R} \iff h \in \text{span}\{\mathbf{1}\} \end{aligned}$$

where, in passing from the first to the second line, we exploited that h is periodic. \square

Claim: $\text{ran}(F_u(0, 0)) = \left\{ z \in C_{\text{per}}(\mathbb{R}) : \int_0^{2\pi} z(x) dx = 0 \right\}$.

Proof: Assume $z \in \text{ran}(F_u(0, 0))$. Then, there exists $w \in C_{\text{per}}^2(\mathbb{R})$ with $z = F_u(0, 0)[w] = w''$ on \mathbb{R} . This yields, however,

$$\int_0^{2\pi} z(x) dx = \int_0^{2\pi} w''(x) dx = w'(2\pi) - w'(0) = 0$$

since w is periodic.

Conversely, assume that $z \in C_{\text{per}}(\mathbb{R})$ with $\int_0^{2\pi} z(x) \, dx = 0$. We try to find $w \in C_{\text{per}}^2(\mathbb{R})$ with $z = F_u(0,0)[w] = w''$ on \mathbb{R} . Integration suggests the ansatz

$$\begin{aligned} w(x) &= \alpha + \beta x + \int_0^x \int_0^y z(t) \, dt dy, & \text{hence} \\ w'(x) &= \beta + \int_0^x z(t) \, dt \end{aligned}$$

with constants $\alpha = w(0)$ and $\beta = w'(0)$. By assumption on z , we have for $x \in \mathbb{R}$

$$w'(x + 2\pi) - w'(x) = \int_0^{2\pi} z(x) \, dx = 0,$$

i.e. w' has the asserted periodicity. Moreover, choosing $\beta := -\frac{1}{2\pi} \int_0^{2\pi} \int_0^y z(t) \, dt dy$, we also have, using that w' is 2π -periodic,

$$\begin{aligned} w(x + 2\pi) - w(x) &= \int_x^{x+2\pi} w'(t) \, dt = \int_0^{2\pi} w'(t) \, dt \\ &= w(2\pi) - w(0) = 2\pi\beta + \int_0^{2\pi} \int_0^y z(t) \, dt dy = 0 \end{aligned}$$

and hence $w \in C_{\text{per}}^2(\mathbb{R})$ with $z = F_u(0,0)[w]$. □

(c) Proof: Finally, we intend to apply the Crandall-Rabinowitz Bifurcation Theorem.

From part (b) we defer that $F_u(0,0): C_{\text{per}}^2(\mathbb{R}) \rightarrow C_{\text{per}}(\mathbb{R})$ is a (1,1)-Fredholm operator, meaning the simplicity assumption (S) from the lecture is satisfied. It remains to check the transversality condition

$$(T) \quad F_{u\lambda}(0,0)[\phi] \notin \text{ran}(F_x(0,0))$$

where $\phi = \mathbf{1}$. We calculate

$$F_{u\lambda}(0,0)[\phi] = F''(0,0)[(\mathbf{1}, 0), (0, 1)] = \mathbf{1},$$

and using (b) find $F_{u\lambda}(0,0)[\phi] \notin \text{ran}(F_x(0,0))$.

An application of Theorem 4.3 then closes the proof. □