Institute for Analysis
PD Dr. Rainer Mandel
M.Sc. Sebastian Ohrem

Problem Sheet 7
Bifurcation Theory
Winter Semester 2022/23

12.12.2022

Recall the assumptions appearing in Chapter 4:

(A) X, Z are Banach spaces and $F \in C^2(\mathbb{R} \times X; Z)$ such that $F(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$.

(S) $L := F_x(\lambda_0, 0)$ is a (1,1)-Fredholm operator.

(T) $F_x(\lambda_0, 0)[\phi] \notin \operatorname{ran}(L)$ where $\ker(L) = \mathbb{R}\phi$.

Problem 19:
Recall the setting of the Crandall-Rabinowitz theorem: Assume (A) and (S), which allows us to decompose $X = \ker(L) \oplus \tilde{X}$ where $\ker(L) = \mathbb{R}\phi$.

(a) Let $\Pi\colon Z \to \operatorname{ran}(L)$ denote a projection onto $\operatorname{ran}(L)$, U be an open neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R}^2$, and $\hat{y} \in C^2(U; \tilde{X})$ be such that $\hat{y}(\lambda_0, 0) = 0$ and

$$\Pi F(\lambda, \hat{y}(\lambda, s) + s\phi) = 0 \text{ for } (\lambda, s) \in U.$$

Show $\hat{y}_\lambda(\lambda_0, 0) = \hat{y}_s(\lambda_0, 0) = 0$ and calculate $\hat{y}_{ss}(\lambda_0, 0)$.

(b) Also let $\psi \in Z'$ such that $\ker(\psi) = \operatorname{ran}(L)$ and define

$$\mathfrak{F}(\lambda, s) := \psi F(\lambda, \hat{y}(\lambda, s) + s\phi) \text{ for } (\lambda, s) \in U.$$

Calculate the derivatives $\mathfrak{F}_\lambda(\lambda_0, 0)$, $\mathfrak{F}_s(\lambda_0, 0)$, $\mathfrak{F}_{ss}(\lambda_0, 0)$, $\mathfrak{F}_{s\lambda}(\lambda_0, 0)$.

Problem 20:
Show that the statement of the Crandall-Rabinowitz theorem in general does not hold for F satisfying (A) where $F_x(\lambda_0, 0)$ is a $(2, 2)$-Fredholm operator.