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Solution to Problem Sheet 7

Bifurcation Theory

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Recall the assumptions appearing in Chapter 4:

(A) X, Z are Banach spaces and $F \in C^2(\mathbb{R} \times X; Z)$ such that $F(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$.

(S) $L := F_x(\lambda_0, 0)$ is a (1,1)-Fredholm operator.

(T) $F_x(\lambda_0, 0)[\phi] \notin \operatorname{ran}(L)$ where $\ker(L) = \mathbb{R}\phi$.

Problem 19:
Recall the setting of the Crandall-Rabinowitz theorem: Assume (A) and (S), which allows us to decompose $X = \ker(L) \oplus \tilde{X}$ where $\ker(L) = \mathbb{R}\phi$.

(a) Let $\Pi \colon Z \to \operatorname{ran}(L)$ denote a projection onto $\operatorname{ran}(L)$, U be an open neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R}^2$, and $\hat{y} \in C^2(U; \tilde{X})$ be such that $\hat{y}(\lambda_0, 0) = 0$ and

$$\Pi F(\lambda, \hat{y}(\lambda, s) + s\phi) = 0 \text{ for } (\lambda, s) \in U. \tag{1}$$

Show $\hat{y}_\lambda(\lambda_0, 0) = \hat{y}_s(\lambda_0, 0) = 0$ and calculate $\hat{y}_{ss}(\lambda_0, 0)$.

(b) Also let $\psi \in Z'$ such that $\ker(\psi) = \operatorname{ran}(L)$ and define

$$\mathfrak{F}(\lambda, s) := \psi F(\lambda, \hat{y}(\lambda, s) + s\phi) \text{ for } (\lambda, s) \in U.$$

Calculate the derivatives $\mathfrak{F}_\lambda(\lambda_0, 0)$, $\mathfrak{F}_s(\lambda_0, 0)$, $\mathfrak{F}_{ss}(\lambda_0, 0)$, $\mathfrak{F}_{s\lambda}(\lambda_0, 0)$.

Solution to problem 19:

(a) *Proof:* Denote the term on the left-hand side of (1) by t. Then we obtain:

$$\begin{aligned}
0 &= \partial_\lambda t = \Pi F_\lambda(\lambda, \hat{y}(\lambda, s) + s\phi) + \Pi F_x(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_\lambda(\lambda, s)], \\
0 &= \partial_s t = \Pi F_x(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_s(\lambda, s) + \phi], \\
0 &= \partial_s^2 t = \Pi F_{xx}(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_s(\lambda, s) + \phi, \hat{y}_s(\lambda, s) + \phi] + \Pi F_x(\lambda, \hat{y}(\lambda, s) + s\phi)[\hat{y}_{ss}(\lambda, s)],
\end{aligned}$$

and in particular

$$\begin{aligned}
0 &= \Pi F_\lambda(\lambda_0, 0) + \Pi F_x(\lambda_0, 0)[\hat{y}_\lambda(\lambda_0, 0)] = 0 + \Pi L \hat{y}_\lambda(\lambda_0, 0), \\
0 &= \Pi F_x(\lambda_0, 0)[\hat{y}_s(\lambda_0, 0) + \phi] = \Pi L \hat{y}_s(\lambda_0, 0) + \Pi L\phi = \Pi L \hat{y}_s(\lambda_0, 0), \\
0 &= \Pi F_{xx}(\lambda_0, 0)[\hat{y}_s(\lambda_0, 0) + \phi, \hat{y}_s(\lambda_0, 0) + \phi] + \Pi L \hat{y}_{ss}(\lambda_0, 0).
\end{aligned}$$

Since $\Pi L|_{\tilde{X}} \colon \tilde{X} \to \operatorname{ran}(L)$ is invertible, we can apply the inverse to the above equations to obtain

$$\begin{aligned}
\hat{y}_\lambda(\lambda_0, 0) &= 0, \\
\hat{y}_s(\lambda_0, 0) &= 0, \\
\hat{y}_{ss}(\lambda_0, 0) &= -(\Pi L|_{\tilde{X}})^{-1} \Pi F_{xx}(\lambda_0, 0)[\phi, \phi].
\end{aligned}$$

We can thus characterize $\hat{y}_{ss}(\lambda_0, 0)$ by saying that it is the unique $\zeta \in \tilde{X}$ with

$$L\zeta = -F_{xx}(\lambda_0, 0)[\phi, \phi]. \qquad \square$$

(b) <u>*Proof:*</u> Similar to part (a) we have

$$\begin{aligned}
\mathfrak{F}_\lambda(\lambda,s) &= \psi F_\lambda(\lambda,\hat{y}(\lambda,s)+s\phi) + \psi F_x(\lambda,\hat{y}(\lambda,s)+s\phi)[\hat{y}_\lambda(\lambda,s)],\\
\mathfrak{F}_s(\lambda,s) &= \psi F_x(\lambda,\hat{y}(\lambda,s)+s\phi)[\hat{y}_s(\lambda,s)+\phi],\\
\mathfrak{F}_{ss}(\lambda,s) &= \psi F_{xx}(\lambda,\hat{y}(\lambda,s)+s\phi)[\hat{y}_s(\lambda,s)+\phi,\hat{y}_s(\lambda,s)+\phi] + \psi F_x(\lambda,\hat{y}(\lambda,s)+s\phi)[\hat{y}_{ss}(\lambda,s)],\\
\mathfrak{F}_{s\lambda}(\lambda,s) &= \psi F_{x\lambda}(\lambda,\hat{y}(\lambda,s)+s\phi)[\hat{y}_s(\lambda,s)+\phi]\\
&\quad + \psi F_{xx}(\lambda,\hat{y}(\lambda,s)+s\phi)[\hat{y}_s(\lambda,s)+\phi,\hat{y}_\lambda(\lambda,s)]\\
&\quad + \psi F_x(\lambda,\hat{y}(\lambda,s)+s\phi)[\hat{y}_{s\lambda}(\lambda,s)+\phi].
\end{aligned}$$

Using $\psi F_x(\lambda_0,0)=\psi L=0$ and the formulas above, we obtain

$$\begin{aligned}
\mathfrak{F}_\lambda(\lambda_0,0) &= 0,\\
\mathfrak{F}_s(\lambda_0,0) &= 0,\\
\mathfrak{F}_{ss}(\lambda_0,0) &= \psi F_{xx}(\lambda_0,0)[\phi,\phi],\\
\mathfrak{F}_{s\lambda}(\lambda_0,0) &= \psi F_{x\lambda}(\lambda_0,0)[\phi].
\end{aligned}$$

□

Problem 20:
Show that the statement of the Crandall-Rabinowitz theorem in general does not hold for F satisfying (A) where $F_x(\lambda_0,0)$ is a $(2,2)$-Fredholm operator.

Solution to problem 20:
We consider a nonlinear modification of the eigenvalue problem:

$$F\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d,\, F(\lambda,x) = Ax - \lambda x - \|x\|^2 Bx$$

where in general $A,B\in\mathbb{R}^{d\times d}$. In the following, we choose $d=2$ and $A=0$.

(a) Let $B=\begin{pmatrix}0&0\\0&0\end{pmatrix}$. Then nontrivial solutions of $F(\lambda,x)=0$ are given by $\lambda=0$, $x\in\mathbb{R}^2\setminus\{0\}$ arbitrary. These solution form a 2-dimensional surface bifurcating from $(0,(0,0))$.

(b) Let $B=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. Then nontrivial solutions of $F(\lambda,x)=0$ are given either by $\lambda=0, x_1=0, x_2\in\mathbb{R}\setminus\{0\}$ arbitrary, or by $\lambda=-x_1^2, x_2=0, x_1\in\mathbb{R}$ arbitrary. These solutions form two distinct curves bifurcating from $(0,(0,0))$.

(c) Let $B=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Then nontrivial solutions of $F(\lambda,x)=0$ are given by $\lambda=0, x_2=0, x_1\in\mathbb{R}\setminus\{0\}$ arbitrary. These solutions form a single curves bifurcating from $(0,(0,0))$.

(d) Let $B=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. Then nontrivial solutions of $F(\lambda,x)=0$ do not exist.

Since nontrivial solutions of $F(\lambda,x)=0$ behave differently in all 4 examples, while the considered problem in each is a smooth, well-behaved perturbation of the eigenvalue problem $Ax=\lambda x$, we expect that the Crandall-Rabinowitz theorem does not generalize to $(2,2)$-Fredholm operators.