Karlsruhe Institute of Technology

Institute for Analysis
PD Dr. Rainer Mandel
M.Sc. Sebastian Ohrem

Problem Sheet 8

Bifurcation Theory

Winter Semester 2022/23

19.12.2022

Problem 21:
For $u \in H^2(\mathbb{R})$ we consider the problem

$$\begin{cases} (x^2u')'(x) - \lambda u(x) = u(x)^3 \text{ on } (1, \mathrm{e}), \\ u(1) = u(\mathrm{e}) = 0. \end{cases} \tag{1}$$

(a) With $X := H^2((1,\mathrm{e})) \cap H_0^1((1,\mathrm{e}))$, $Z := L^2((1,\mathrm{e}))$ rephrase (1) as

$$Lu - \lambda u = N(u) \text{ where } u \in X$$

with appropriate $L \in \mathcal{L}(X;Z)$ and $N \in C^\infty(X;Z)$ such that $N'(0) = 0$.

(b) Show that $\lambda_0 := -1/4 - \pi^2$ is a simple eigenvalue of L and calculate the associated eigenfunction ϕ.
Hint: Use the Ansatz $u(x) = w(\log(x))$ to find the eigenfunction.[1]

(c) Show that $L - \lambda_0 I \colon X \to Z$ is a Fredholm operator of index 0.

(d) Show that L is symmetric, i.e. that

$$\langle Lu|v\rangle_{L^2} = \langle u|Lv\rangle_{L^2}$$

holds for $u, v \in X$ and conclude that $\operatorname{ran}(L - \lambda_0 I) = \phi^\perp = \ker(\langle \cdot |\phi\rangle_{L^2})$.

(e) Show that there exists a curve of nontrivial solutions $(\hat\lambda(s), \hat u(s))$ of (1) bifurcating from $(\lambda_0, 0)$.

(f) Calculate $\hat\lambda'(0)$ and $\hat\lambda''(0)$.

Problem 22:
Consider the stationary Gross-Pitaevskii equation

$$-u''(x) + V(x)u(x) + \sigma u(x)^3 = \lambda u(x) \tag{2}$$

for $u \in H^2(\mathbb{R})$, where $\sigma \in \{\pm 1\}$, $\lambda \in \mathbb{R}$ and $V \in L_0^\infty(\mathbb{R})$, which is defined as $L_0^\infty(\mathbb{R}) := \overline{C_c^\infty(\mathbb{R})}^{\|\cdot\|_\infty}$.

(a) Let

$$M_V \colon H^2(\mathbb{R}) \to L^2(\mathbb{R}), u \mapsto Vu, \qquad L \colon H^2(\mathbb{R}) \to L^2(\mathbb{R}), u \mapsto -u'' + Vu,$$

Show that M_V is compact and conclude that $L - \lambda I \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Fredholm operator of index 0 for $\lambda \in (-\infty, 0)$.
It is known that $-\partial_x^2 \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ has spectrum $[0,\infty)$.

(b) Let $\lambda_0 < 0$ be a simple eigenvalue of $-\partial_x^2 + V(x) \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$.
Show that there is a curve $(\hat\lambda(s), \hat u(s))$ of nontrivial solutions to (2) bifurcating from $(\lambda_0, 0)$.

(c) Determine the direction of bifurcation (i.e. whether $\hat\lambda(s) > \lambda_0$ or $\hat\lambda(s) < \lambda_0$ for $s \neq 0$).

[1] This ODE is of the form $\sum_{k=0}^K a_k x^k u^{(k)}(x) = 0$, called Euler's equation. Using the Ansatz $u(x) = w(\log(x))$ results in an ODE with constant coefficients for w!