Institute for Analysis
PD Dr. Rainer Mandel
M.Sc. Sebastian Ohrem

Solution to Problem Sheet 8

Bifurcation Theory

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Problem 21:
For $u \in H^2((1,\mathrm{e}))$ we consider the problem

$$(1) \qquad \begin{cases} (x^2u')'(x) - \lambda u(x) = u(x)^3 \text{ on } (1,\mathrm{e}), \\ u(1) = u(\mathrm{e}) = 0. \end{cases}$$

(a) With $X := H^2((1,\mathrm{e})) \cap H_0^1((1,\mathrm{e}))$, $Z := L^2((1,\mathrm{e}))$ rephrase (1) as

$$Lu - \lambda u = N(u) \text{ where } u \in X$$

with appropriate $L \in \mathcal{L}(X;Z)$ and $N \in C^\infty(X;Z)$ such that $N'(0) = 0$.

(b) Show that $\lambda_0 := -1/4 - \pi^2$ is a simple eigenvalue of L and calculate the associated eigenfunction ϕ. *Hint:* Use the Ansatz $u(x) = w(\log(x))$ to find the eigenfunction.[1]

(c) Show that $L - \lambda_0 I : X \to Z$ is a Fredholm operator of index 0.

(d) Show that L is symmetric, i.e. that

$$\langle Lu|v\rangle_{L^2} = \langle u|Lv\rangle_{L^2}$$

holds for $u, v \in X$ and conclude that $\operatorname{ran}(L - \lambda_0 I) = \phi^\perp = \ker(\langle \cdot|\phi\rangle_{L^2})$.

(e) Show that there exists a curve of nontrivial solutions $(\hat\lambda(s), \hat u(s))$ of (1) bifurcating from $(\lambda_0, 0)$.

(f) Calculate $\hat\lambda'(0)$ and $\hat\lambda''(0)$.

Solution to problem 21:

(a) We choose $Lu = (x^2u')'$ as well as $Nu = u^3$. Note that the embedding $H^2(\mathbb{R}) \hookrightarrow L^6(\mathbb{R})$ is continuous, so that N is well-defined. Furthermore, L is clearly linear, and it is bounded since we can estimate

$$\|Lu\|_2 = \left\|x^2u'' + 2xu'\right\|_2 \le \mathrm{e}^2\|u''\|_2 + 2\mathrm{e}\|u'\|_2 \le (\mathrm{e}^2 + 2\mathrm{e})\|u\|_{H^2}$$

To show that N is infinitely many times differentiable and

$$N'(u)[h] = 3u^2h, \quad N''(u)[h_1,h_2] = 6uh_1h_2, \quad N'''(u)[h_1,h_2,h_3] = 6h_1h_2h_3, \quad N'''' = 0$$

we use that $\|h\|_6 \le C\|h\|_{H^2}$ for $u \in H^2((1,\mathrm{e}))$ for some $C > 0$. Then we estimate

$$N(u+h) - N(u) = 3u^2h + 3uh^2 + h^3 = 3u^2h + \mathcal{O}(\|u\|_5\|h\|_6^2 + \|h\|_6^3) = 3u^2h + \mathcal{O}(\|h\|_{H^2}^2),$$

as $h \to 0$. Similarly, we estimate

$$N'(u+h)[h_2] - N'(u)[h_2] = 6uhh_2 + 3h^2h_2 = 6uhh_2 + \|h_2\|_{H^2}\mathcal{O}(\|h\|_{H^2}^2)$$

as $h \to 0$. Finally, we have

$$N''(u+h)[h_2,h_3] - N''(u)[h_2,h_3] = 6hh_2h_3 = N'''(u)[h,h_2,h_3],$$

and $N'''' = 0$ is clear since N''' is constant, i.e. independent of u.

[1]This ODE is of the form $\sum_{k=0}^K a_kx^ku^{(k)}(x) = 0$, called Euler's equation. Using the Ansatz $u(x) = w(\log(x))$ results in an ODE with constant coefficients for w!

(b) *Proof:* Using the Ansatz $u(x) = w(\log(x))$, we have

$$u'(x) = \frac{w'(\log(x))}{x}, \qquad u''(x) = \frac{w''(\log(x)) - w'(\log(x))}{x^2}$$

and therefore

$$(x^2u')'(x) - \lambda u(x) = x^2u''(x) + 2xu'(x) - \lambda u(x) = w''(\log(x)) + w'(\log(x)) - \lambda w(\log(x))$$

So we have to show that $H^2(0,1) \cap H_0^1(0,1)$-solutions to

$$\begin{cases} w'' + w' + (1/4 + \pi^2)w = 0 \text{ on } (0,1), \\ w(0) = w(1) = 0 \end{cases} \tag{2}$$

span a space of dimension 1. Observe first that any solution is already a classical solution:

Claim: Any solution w to (2) lies in $C^2([0,1])$, satisfies $w'' + w' + (1/4 + \pi^2)w = 0$ pointwise as well as $w(0) = w(1) = 0$.

Proof: Let w be a solution. Since the embedding $H^2((0,1)) \hookrightarrow C^1([0,2])$ is continuous, we have $w \in C^1([0,1])$. Furthermore, $w'' = -w' - (1/4 + \pi^2)w$ is continuous. It is a well-known fact that from this it follows that $w \in C^2([0,1])$. It is also known that $w \in C^2([0,1])$ satisfies $w \in H_0^1([0,1])$ if and only if $w(0) = w(1) = 0$. □

This allows us to use classical theory for constant coefficient ODEs. Consider the characteristic polynomial $p(\lambda) = \lambda^2 + \lambda + (1/4 + \pi^2)$, which has zeros $-1/2 \pm \mathrm{i}\pi$. Thus the general solution of $w''(y) + w'(y) + (1/4 + \pi^2)w(y) = 0$ is spanned by

$$\mathrm{e}^{(-1/2+\mathrm{i}\pi)y}, \quad \mathrm{e}^{(-1/2-\mathrm{i}\pi)y}$$

which has real-valued analogue

$$\mathrm{e}^{-y/2}\sin(\pi y), \quad \mathrm{e}^{-y/2}\cos(\pi y).$$

We see that the subspace of solutions satisfying $w(0) = w(1) = 0$ is indeed one-dimensional and spanned by $w(y) = \mathrm{e}^{-y/2}\sin(\pi y)$. So $-1/4 - \pi^2$ is a simple eigenvalue and the associated eigenfunction is (up to scalars) given by

$$u(x) = w(\log(x)) = \frac{\sin(\pi \log(x))}{\sqrt{x}}.$$

In the following, we denote this function by $\phi(x)$. □

(c) *Proof:* We have previously seen in the exercise classes that the laplacian $L \colon X \to Z$ is invertible. As the embedding $X \hookrightarrow Z$ is compact, $L - \lambda_0 I \colon X \to Z$ is Fredholm of index 0 as a compact perturbation of an invertible operator. □

(d) *Proof:* To show that L is symmetric, let u, v in X. Then

$$\langle Lu|v\rangle_{L^2} = \int_1^{\mathrm{e}} (x^2u')'v \,\mathrm{d}x = -\int_1^{\mathrm{e}} (x^2u')v' \,\mathrm{d}x = -\int_1^{\mathrm{e}} x^2u'v' \,\mathrm{d}x$$

where for the second equality we have used that

$$\int_\Omega f'g \,\mathrm{d}x = -\int_\Omega fg' \,\mathrm{d}x \quad \text{ for } f \in H^1(\Omega), g \in H_0^1(\Omega).$$

Similarly, we get $\langle u|Lv\rangle_{L^2} = -\int_1^{\mathrm{e}} x^2u'v' \,\mathrm{d}x$, showing that L is symmetric.

Since L is symmetric, also $L - \lambda_0 I$ is symmetric. From the exercise class we know that the relation $\operatorname{ran}(L - \lambda_0 I) \perp \ker(L - \lambda_0 I)$ holds, i.e. $\operatorname{ran}(L - \lambda_0 I) \subseteq \ker(L - \lambda_0 I)^\perp = \phi^\perp$. Since both spaces have codimension 1, we conclude $\operatorname{ran}(L - \lambda_0 I) = \phi^\perp$. □

(e) *Proof:* We intend to apply the Crandall-Rabinowitz theorem. Let

$$F\colon \mathbb{R}\times X \to Z, F(\lambda,u) = (x^2u')' - \lambda u - u^3$$

By the observations in part (a) plus the chain rule we have $F\in C^\infty(\mathbb{R}\times X;Z)$ as well as

$$F_u(\lambda_0,0)[h] = (x^2h')' F_{u\lambda}(\lambda_0,0)[h] = -hF_{xx}(\lambda_0,0) = 0 F_{xxx}(\lambda_0,0)[h_1,h_2,h_3] = -6h_1h_2h_3.$$

Clearly $F(\lambda,0)=0$ holds for $\lambda\in\mathbb{R}$. By part (b) the simplicity assumption (S) is satisfied, and (T) holds since

$$F_{u\lambda}(\lambda_0,0)[\phi] = -\phi \notin \phi^\perp = \operatorname{ran}(F_u(\lambda_0,0)).$$

Thus we may employ the Crandall-Rabinowitz theorem to obtain a curve $(\hat\lambda(s),\hat u(s))$ of nontrivial solutions to (1) bifurcating from $(\lambda_0,0)$. □

(f) *Proof:* Using the derivative formulas from Corollary 4.6 we get

$$\hat\lambda'(0) = 0$$

as well as

$$\hat\lambda''(0) = -\frac{1}{3}\frac{6\langle\phi^3|\phi\rangle}{\langle-\phi|\phi\rangle} = 2\frac{\int_1^{\mathrm{e}}\phi^4\,\mathrm{d}x}{\int_1^{\mathrm{e}}\phi^2\,\mathrm{d}x}$$

By substituting $y=\log(x)$, we can calculate

$$\int_1^{\mathrm{e}}\phi(x)^2\,\mathrm{d}x = \int_1^{\mathrm{e}}\frac{\sin(\pi\log(x))^2}{x}\,\mathrm{d}x = \int_0^1\sin(\pi y)^2\,\mathrm{d}y = \frac{1}{2},$$

$$\int_1^{\mathrm{e}}\phi(x)^4\,\mathrm{d}x = \int_1^{\mathrm{e}}\frac{\sin(\pi\log(x))^4}{x^2}\,\mathrm{d}x = \int_0^1\mathrm{e}^{-y}\sin(\pi y)^4\,\mathrm{d}y = \frac{24(\mathrm{e}-1)\pi^4}{\mathrm{e}(1+4\pi^2)(1+16\pi^2)},$$

so

$$\hat\lambda''(0) = \frac{96(\mathrm{e}-1)\pi^4}{\mathrm{e}(1+4\pi^2)(1+16\pi^2)}.$$

To calculate the last integral by hand, one can for example write $\sin(z)=\frac{1}{2\mathrm{i}}\big(\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z}\big)$, yielding

$$\mathrm{e}^{-y}\sin(\pi y)^4 = \frac{\mathrm{e}^{-y}}{16}\big(\mathrm{e}^{4\mathrm{i}\pi y} - 4\mathrm{e}^{2\mathrm{i}\pi y} + 6 - 4\mathrm{e}^{-2\mathrm{i}\pi y} + \mathrm{e}^{-4\mathrm{i}\pi y}\big) = \frac{\mathrm{e}^{-y}}{16}(2\cos(4\pi y) - 8\cos(2\pi y) + 6).$$

Using

$$\begin{aligned}\int_0^1\mathrm{e}^{-y}\cos(2\pi k y)\,\mathrm{d}y &= \operatorname{Re}\bigg(\int_0^1\mathrm{e}^{(-1+2\pi\mathrm{i}k)y}\,\mathrm{d}y\bigg) = \operatorname{Re}\bigg(\frac{\mathrm{e}^{-1+2\pi\mathrm{i}k}-1}{-1+2\pi\mathrm{i}k}\bigg)\\ &= \operatorname{Re}\bigg(\frac{(\mathrm{e}^{-1}-1)(-1-2\pi\mathrm{i}k)}{1+4\pi^2k^2}\bigg) = \frac{\mathrm{e}-1}{\mathrm{e}(1+4\pi^2k^2)}\end{aligned}$$

we finally obtain

$$\begin{aligned}\int_0^1\mathrm{e}^{-y}\sin(\pi y)^4\,\mathrm{d}y &= \frac{\mathrm{e}-1}{16\mathrm{e}}\bigg(\frac{2}{1+16\pi^2} - \frac{8}{1+4\pi^2} + 6\bigg)\\ &= \frac{(\mathrm{e}-1)\big(2(1+4\pi^2) - 8(1+16\pi^2) + 6(1+4\pi^2)(1+16\pi^2)\big)}{16\mathrm{e}(1+4\pi^2)(1+16\pi^2)}\\ &= \frac{24(\mathrm{e}-1)\pi^4}{\mathrm{e}(1+4\pi^2)(1+16\pi^2)}.\end{aligned}$$

□

Problem 22:
Consider the stationary Gross-Pitaevskii equation

$$-u''(x) + V(x)u(x) + \sigma u(x)^3 = \lambda u(x) \tag{3}$$

for $u \in H^2(\mathbb{R})$, where $\sigma \in \{\pm 1\}$, $\lambda \in \mathbb{R}$ and $V \in L^\infty_0(\mathbb{R})$, which is defined as $L^\infty_0(\mathbb{R}) := \overline{C_c^\infty(\mathbb{R})}^{\|\cdot\|_\infty}$.

(a) Let

$$M_V \colon H^2(\mathbb{R}) \to L^2(\mathbb{R}), u \mapsto Vu, \qquad L \colon H^2(\mathbb{R}) \to L^2(\mathbb{R}), u \mapsto -u'' + Vu,$$

Show that M_V is compact and conclude that $L - \lambda I \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Fredholm operator of index 0 for $\lambda \in (-\infty, 0)$.

It is known that $-\partial_x^2 \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ has spectrum $[0, \infty)$.

(b) Let $\lambda_0 < 0$ be a simple eigenvalue of $-\partial_x^2 + V(x) \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Show that there is a curve $(\hat{\lambda}(s), \hat{u}(s))$ of nontrivial solutions to (3) bifurcating from $(\lambda_0, 0)$.

(c) Determine the direction of bifurcation (i.e. whether $\hat{\lambda}(s) > \lambda_0$ or $\hat{\lambda}(s) < \lambda_0$ for $s \neq 0$).

Solution to problem 22:

(a) *Claim:* $M_V \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is compact.
Proof: First let $V \in C_c^\infty(\mathbb{R})$ and set $K := \operatorname{supp}(V)$. We then may decompose

$$M_V \colon H^2(\mathbb{R}) \xrightarrow{\text{restriction}} H^2(K) \xrightarrow[\text{(compact)}]{\text{embedding}} L^2(K) \xrightarrow[\text{with } V]{\text{multiplication}} L^2(K) \xrightarrow[\text{by zero}]{\text{continuation}} L^2(\mathbb{R})$$

where all maps appearing are bounded linear maps, and one of them is compact, from which it follows that also M_V is compact, see e.g. Problem 14 (d).

To complete the argument for general $V \in L^\infty_0(\mathbb{R})$, let $V_n \in C_c^\infty(\mathbb{R})$ be a sequence such that $V_n \to V$ in $L^\infty(\mathbb{R})$. Since $\|Vu - V_n u\|_2 \leq \|V - V_n\|_{L^\infty(\mathbb{R})} \|u\|_{L^2(\mathbb{R})} \leq \|V - V_n\|_{L^\infty(\mathbb{R})} \|u\|_{H^2(\mathbb{R})}$, we have

$$\|M_V - M_{V_n}\|_{\mathcal{L}(H^2(\mathbb{R}); L^2(\mathbb{R}))} \leq \|V - V_n\|_\infty,$$

so that $M_{V_n} \to M_V$ in $\mathcal{L}(H^2(\mathbb{R}); L^2(\mathbb{R}))$. Compactness of M_V now follows from compactness of the M_{V_n} using Problem 14 (c). □

Claim: $L - \lambda I$ is a Fredholm operator of Index 0 for $\lambda \in (-\infty, 0)$.
Proof: Let $\lambda \in (-\infty, 0)$. As λ does not lie in the spectrum of $-\partial_x^2$, the operator $-\partial_x^2 - \lambda I \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is invertible. Since M_V is compact, it follows that $L - \lambda I = (-\partial_x^2 - \lambda I) + M_V$ is a Fredholm operator of index 0 as a compact perturbation of an invertible operator. □

(b) *Proof:* We intend to apply the Crandall-Rabinowitz theorem. Set $X := H^2(\mathbb{R})$, $Z := L^2(\mathbb{R})$ and

$$F \colon \mathbb{R} \times X \to Z, F(\lambda, u) = -u'' + V(x)u - \lambda u + \sigma u^3.$$

We note that $u \in C^{1,1/2}(\mathbb{R})$ by the Sobolev embedding, so in particular $u \in L^6(\mathbb{R})$ and thus $u^3 \in L^2(\mathbb{R})$, i.e. F is well-defined.

One can check that F is infinitely many times differentiable and

$$\begin{aligned}
&F'(\lambda, u)[(\mu, h)] = -h'' + V(x)h - \lambda h - \mu u + 3\sigma u^2 h \\
&F''(\lambda, u)[(\mu_1, h_1), (\mu_2, h_2)] = -\mu_1 h_2 - \mu_2 h_1 + 6\sigma u h_1 h_2 \\
&F'''(\lambda, u)[(\mu_1, h_1), (\mu_2, h_2), (\mu_3, h_3)] = 6\sigma h_1 h_2 h_3 \\
&F''''(\lambda, u) = 0
\end{aligned}$$

We also have $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

Next we check the simplicity assumption (S). Observe that $L - \lambda_0 I = F_u(\lambda_0, 0)$, so that by part (a), $F_u(\lambda_0, 0)$ is a Fredholm operator of index 0. Furthermore, $F_u(\lambda_0, 0)$ has one-dimensional kernel by assumption, wherefore $F_u(\lambda_0, 0)$ is a $(1,1)$-Fredholm operator.

It remains to check the transversality condition (T), i.e.

$$F_{u\lambda}(\lambda_0, 0)[\phi] \notin \operatorname{ran}(F_u(\lambda_0, 0))$$

where $\mathbb{R}\phi = \ker(F_x(\lambda_0, 0))$. The left-hand side is given by

$$F_{u\lambda}(\lambda_0, 0)[\phi] = -\phi.$$

To calculate the right-hand side, we use that $L - \lambda_0 I$ is symmetric, i.e. that

$$\langle (L - \lambda_0 I)u | v \rangle_{L^2(\mathbb{R})} = \langle u | (L - \lambda_0 I)v \rangle_{L^2(\mathbb{R})} \tag{4}$$

holds for $u, v \in H^2(\mathbb{R})$, which is obtained by applying integration by parts twice. From (4) it follows by simple calculation that $\mathbb{R}\phi = \ker(L - \lambda_0 I) \perp \operatorname{ran}(L - \lambda_0 I)$. Since $\operatorname{ran}(L - \lambda_0 I)$ is closed and has codimension 1, we conclude $\operatorname{ran}(L - \lambda_0 I) = \phi^\perp$. In particular, $-\phi \notin \phi^\perp$ so that the transversality condition holds.

We are thus able to apply the Crandall-Rabinowitz theorem which yields the existence of curve $(\hat{\lambda}(s), \hat{u}(s))$ of nontrivial solutions to (3) bifurcating from $(\lambda_0, 0)$. ☐

Remark: Actually, $L - \lambda_0 I$ is not only symmetric, but self-adjoint. For self-adjoint operators T, the relations $\ker(T) = \operatorname{ran}(T)^\perp$ and $\overline{\operatorname{ran}(T)} = \ker(T)^\perp$ are always satisfied.

(c) *Proof:* We calculate $\hat{\lambda}'(0)$ and (as this will be zero) also $\hat{\lambda}''(0)$ using the formulas provided by Corollary 4.6.

Set $\psi(f) = \langle f | \phi \rangle$ for $f \in Z$. Recall that

$$\begin{aligned} &F_{u\lambda}(\lambda_0, 0)[\phi] = -\phi, \\ &F_{uu}(\lambda_0, 0) = 0, \\ &F_{uuu}(\lambda_0, 0)[\phi, \phi, \phi] = \sigma \phi^3, \end{aligned}$$

so

$$\hat{\lambda}'(0) = -\frac{1}{2} \frac{\psi(F_{uu}(\lambda_0, 0)[\phi, \phi])}{\psi(F_{u\lambda}(\lambda_0, 0))} = 0,$$

which allows us to calculate

$$\hat{\lambda}''(0) = -\frac{1}{3} \frac{\psi(F_{uuu}(\lambda_0, 0)[\phi, \phi, \phi] - 3F_{xx}(\lambda_0, 0)[?, ?])}{\psi(F_{u\lambda}(\lambda_0, 0))} = 2\sigma \frac{\langle \phi^3 | \phi \rangle}{\langle \phi | \phi \rangle} = 2\sigma \frac{\int_{\mathbb{R}} \phi^4 \, \mathrm{d}x}{\int_{\mathbb{R}} \phi^2 \, \mathrm{d}x}$$

We conclude that if $\sigma = +1$, bifurcation from the left occurs (i.e. $\hat{\lambda}(s) > \lambda_0$ for $s \neq 0$), whereas for $\sigma = -1$ bifurcation from the right occurs. ☐