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Solution to Problem Sheet 9

Bifurcation Theory

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Problem 23 (Bending an elastic rod):
The bending of an elastic rod can be described be the boundary value problem

$$
\begin{cases} u'' + \lambda \sin(u) = 0 & \text{in } (0, 2\pi), \\ u'(0) = u'(2\pi) = 0. \end{cases} \tag{1}
$$

Find all bifurcation points for problem (1). Sketch the bifurcation diagram near each bifurcation point $(0, \lambda_j)$ with $\lambda_j > 0$ using the formulas in Corollary 4.6.

Solution to problem 23:
We consider the Banach spaces $X := \{u \in C^2([0, 2\pi]) : u'(0) = u'(2\pi) = 0\}$, which is a closed subspace of $C^2([0, 2\pi])$, and $Z := C([0, 2\pi])$. We then define the mapping

$$
F : \mathbb{R} \times X \to Z, \quad F(\lambda, u) := u'' + \lambda \sin(u).
$$

We note that $F(\lambda, 0) = 0$ for every $\lambda \in \mathbb{R}$.

Similar to Problem 18, one can shows that F is three times continuously Fréchet differentiable. The derivatives are given by

$$
\begin{aligned}
&F_u(\lambda, u)[h] = v_1'' + \lambda \cos(u)h, && F_{u\lambda}(\lambda, u)[h] = \cos(u)h, \\
&F_{uu}(\lambda, u)[h_1, h_2] = -\lambda \sin(u)h_1h_2, && F_{uuu}(\lambda, u)[h_1, h_2, h_3] = -\lambda \cos(u)h_1h_2h_3
\end{aligned}
$$

for $u, h_1, h_2, h_3 \in X$ and $\lambda \in \mathbb{R}$.

We will first characterize bifurcation points using the Crandall-Rabinowitz Theorem. Then we will use the bifurcation formulae to extract further information.

Step 1. *Claim:* $\ker F_u(\lambda, 0) \neq \{0\}$ if and only if $\lambda = \frac{1}{4}n^2$ for some $n \in \mathbb{N}_0$. Moreover, for $n \in \mathbb{N}_0$ we have

$$
\ker F_u(\lambda_n, 0) = \mathbb{R}\phi_n \qquad \text{where} \qquad \phi_n(x) = \cos\Big(\frac{n}{2} \cdot\Big). \tag{2}
$$

Proof: By definition we have

$$
w \in \ker F_u(\lambda, 0) \qquad \Longleftrightarrow \qquad w \in C^2([0, 2\pi]), \begin{cases} w'' + \lambda w = 0 & \text{in } (0, 2\pi), \\ w'(0) = w'(2\pi) = 0. \end{cases}
$$

The general solution of the ODE is given by $w = \alpha w_1 + \beta w_2$ with

$$
\begin{aligned}
&w_1(x) = \sin(\sqrt{\lambda}x), \quad w_2(x) = \cos(\sqrt{\lambda}x) && \text{if } \lambda > 0, \\
&w_1(x) = x, \qquad\qquad\; w_2(x) = 1 && \text{if } \lambda = 0, \\
&w_1(x) = \sinh(\sqrt{-\lambda}x), w_2(x) = \cosh(\sqrt{\lambda}x) && \text{if } \lambda < 0,
\end{aligned}
$$

noting that $w_1(0) = 0, w_1'(0) = 1, w_2(0) = 1, w_2'(0) = 0$. As $w'(0) = \alpha$, we see that $F_u(\lambda, 0)w = 0$ admits nontrivial solutions w if and only if $w_2'(2\pi) = 0$, which holds if and only if $\lambda = \lambda_n$ for some $n \in \mathbb{N}_0$. In this case the general solution is given by $w(x) = \beta w_2(x) = \beta \cos(\sqrt{\lambda_n}x) = \beta\phi_n(x)$ with $\beta \in \mathbb{R}$. ☐

Step 2. *Claim:* $\operatorname{ran}(F_u(\lambda,0)) \neq Z$ if and only if $\lambda = \lambda_n$ for some $n \in \mathbb{N}_0$. In this case, we have

$$\operatorname{ran}(F_u(\lambda_n,0)) = \left\{ z \in Z \colon \int_0^{2\pi} z(x)\phi_n(x)\,\mathrm{d}x = 0 \right\} =: R.$$

Proof: Let $\lambda \in \mathbb{R}, z \in Z$. Recall:

$$(3) \qquad F_u(\lambda,0)w = z \qquad \Longleftrightarrow \qquad \begin{cases} w'' + \lambda w = z & \text{in } (0,2\pi), \\ w'(0) = w'(2\pi) = 0 \end{cases}$$

We first consider the (inhomogeneous) initial value problem

$$\begin{cases} w'' + \lambda w = z & \text{in } (0,2\pi), \\ w(0) = 0, w'(0) = 0. \end{cases}$$

This problem admits a unique solution $w_z \in C^2([0,2\pi])$ by the Picard-Lindelöf theorem, and the general solution of $w'' + \lambda w = z$ is then given by $w = w_z + \alpha w_1 + \beta w_2$.

Case 1: Let $\lambda \neq \lambda_n$. Then $w_2'(2\pi) \neq 0$. Note that $w = w_z + \alpha w_1 + \beta w_2$ solves (3) if and only if

$$0 = w'(0) = \alpha \text{ and } 0 = w'(2\pi) = w_z'(2\pi) + \alpha w_1'(2\pi) + \beta w_2'(2\pi).$$

Hence with $\alpha = 0$, $\beta = -\frac{w_z'(2\pi)}{w_2'(2\pi)}$ we have $w \in X$ and $F_u(\lambda,0)w = z$. As $z \in Z$ was arbitrary, $F_u(\lambda,0)$ is surjective.

Case 2: Now let $\lambda = \lambda_n$. Then $w_2'(2\pi) = 0$. As before we have $z \in \operatorname{ran}(F_u(\lambda_n,0))$ if and only if $w = w_z + \alpha w_1 + \beta w_2 \in X$ for some $\alpha, \beta \in X$, or equivalently $w'(0) = w'(2\pi) = 0$. Since

$$w'(0) = w'(2\pi) = 0 \iff \alpha = 0, w_z'(2\pi) = 0$$

(independent of β), we have to check $w_z'(2\pi) = 0$. We calculate

$$\begin{aligned} \int_0^{2\pi} z\phi_n\,\mathrm{d}x &= \int_0^{2\pi} (w_z'' + \lambda_n w_z)\phi_n\,\mathrm{d}x \\ &= [w_z'\phi_n]_0^{2\pi} + \int_0^{2\pi} -w_z'\phi_n' + \lambda_n w_z\phi_n\,\mathrm{d}x \\ &= [w_z'\phi_n - w_z\phi_n']_0^{2\pi} + \int_0^{2\pi} w_z\phi_n'' + \lambda_n w_z\,\mathrm{d}x \\ &= w_z'(2\pi)\phi_n(2\pi), \end{aligned}$$

where we have used $\phi_n'(0) = \phi_n'(2\pi) = w_z'(0) = 0$ and $F_u(\lambda_n,0)[\phi_n] = 0$. As $\phi_n(2\pi) \neq 0$, we thus have shown

$$\int_0^{2\pi} z\phi_n\,\mathrm{d}x = 0 \iff w_z'(2\pi) = 0 \iff z \in \operatorname{ran}(F_u(\lambda_n,0)). \qquad \square$$

Step 3. *Claim:* $(0,\lambda)$ is a bifurcation point if and only if $\lambda = \lambda_n$ for some $n \in \mathbb{N}_0$.

Case 1: If $\lambda \neq \lambda_n$, then by the previous two steps we have shown that $F_u(\lambda,0)$ is bijective, and thus invertible by the bounded inverse theorem. Proposition 3.9 shows that $(\lambda,0)$ is not a Bifurcation point of (1).

Case 2: Now let $\lambda = \lambda_n$ for some $n \in \mathbb{N}_0$. We intend to apply the Crandall-Rabinowitz theorem. We have shown that F is C^3, and the simplicity assumption (S) follows from steps 1 and 2. Since

$$F_{u\lambda}(\lambda_n,0)[\phi_n] = \phi_n \notin \operatorname{ran}(F_u(\lambda_n,0))$$

by Step 2, also the transversality condition (T) holds. The Crandall-Rabinowitz theorem can be applied and shows that $(\lambda_n,0)$ is a bifurcation point.

Step 4. Finally, we apply the bifurcation formulae of Corollary 4.7. Let $n \in \mathbb{N}$ and choose

$$\psi(z) = \int_0^{2\pi} z\phi_n \,\mathrm{d}x$$

Since $F_{uu}(\lambda_n, 0) = 0$, we obtain

$$\begin{aligned} \hat{\lambda}'(0) &= 0, \\ \hat{\lambda}''(0) &= -\frac{1}{3}\frac{\psi(F_{uuu}(\lambda_n,0)[\phi_n,\phi_n,\phi_n])}{\psi(F_{u\lambda}(\lambda_n,0)[\phi_n])} \\ &= -\frac{1}{3}\frac{\int_0^{2\pi} -\lambda_n\phi_n^3\cdot\phi_n \,\mathrm{d}x}{\int_0^{2\pi}\phi_n\cdot\phi_n\,\mathrm{d}x} \\ &= \frac{n^2}{16} > 0. \end{aligned}$$

where for the last step we have used $\int_0^{2\pi}\cos(\frac{n}{2}x)^2\,\mathrm{d}x = \pi$ and $\int_0^{2\pi}\cos(\frac{n}{2}x)^4\,\mathrm{d}x = \frac{3}{4}\pi$.

Problem 24 (Bifurcation from ∞):
Consider the nonlinear system

$$\begin{cases} (1-\lambda)x_1 + \frac{x_2}{x_1^2+x_2^2} = 0, \\ (1-2\lambda)x_2 + \frac{x_1}{x_1^2+x_2^2} = 0. \end{cases}$$

Show that bifurcation from ∞ occurs for $\lambda_0 = 1$ and $\lambda_0 = \frac{1}{2}$.

Solution to problem 24:

Proof: Here, we will only consider the case $\lambda_0 = 1$. The case $\lambda_0 = \frac{1}{2}$ can be treated analogously.

Solving the problem is equivalent to finding zeros of the function

$$F : \mathbb{R}\times(\mathbb{R}^2\setminus\{0\}) \to \mathbb{R}^2, \quad F(\lambda, x_1, x_2) := \begin{pmatrix} (1-\lambda)x_1 + \frac{x_2}{x_1^2+x_2^2} \\ (1-2\lambda)x_2 + \frac{x_1}{x_1^2+x_2^2} \end{pmatrix}$$

which is of the form

$$F(\lambda, x) = L(\lambda)x + R(x) \qquad \text{with} \qquad L(\lambda) = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-2\lambda \end{pmatrix}, R(x) = \begin{pmatrix} \frac{x_2}{x_1^2+x_2^2} \\ \frac{x_1}{x_1^2+x_2^2}. \end{pmatrix}$$

Then

$$L(\lambda_0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \ker(L(\lambda_0)) = \mathbb{R}\begin{pmatrix}1\\0\end{pmatrix}, \quad \operatorname{ran}(L(\lambda_0)) = \mathbb{R}\begin{pmatrix}0\\1\end{pmatrix}.$$

Moreover with $\phi := \begin{pmatrix}1\\0\end{pmatrix}$, we have

$$L'(\lambda_0)\phi = \begin{pmatrix}-1\\0\end{pmatrix} \notin \operatorname{ran}(L(\lambda_0)).$$

Hence conditions (S) and (T) of Theorem 5.2 are fulfilled.

Now we check condition (R). We have

$$\mathcal{R}(\lambda, x, s) = sR(\tfrac{x}{s}) = s^2\begin{pmatrix} \frac{x_2}{x_1^2+x_2^2} \\ \frac{x_1}{x_1^2+x_2^2}. \end{pmatrix} = s^2R(x)$$

As $R \in C^\infty(\mathbb{R}^2\setminus\{0\}, \mathbb{R}^2)$, we have $\mathcal{R} \in C^\infty(\mathbb{R}\times(\mathbb{R}^2\setminus\{0\})\times\mathbb{R};\mathbb{R}^2)$ and $\mathcal{R}(\lambda_0, \phi, 0) = 0, \mathcal{R}'(\lambda_0,\phi,0) = 0$. So condition (R) is fulfilled. Using Theorem 5.2, we find that bifurcation at ∞ occurs at $\lambda_0 = 1$. $\square$