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Claim: The derivative of the exponential map is given by

$$\exp'(A)[B] = \int_0^1 \mathrm{e}^{sA} B \mathrm{e}^{(1-s)A} \,\mathrm{d}s$$

Proof: Recall $\exp(A) = \sum_{k=0}^\infty \frac{1}{k!} A^k$. The derivative of $A \mapsto A^k$ in direction B is given by $\sum_{n=1}^k A^{n-1} B A^{k-n}$. Hence we obtain

$$\exp'(A)[B] = \sum_{k=1}^\infty \frac{1}{k!} \sum_{n=1}^k A^{n-1} B A^{k-n}.$$

Now we calculate

$$\begin{aligned}
\int_0^1 \mathrm{e}^{sA} B \mathrm{e}^{(1-s)A} \,\mathrm{d}s &= \int_0^1 \sum_{k=0}^\infty \frac{s^k}{k!} A^k B \sum_{n=0}^\infty \frac{(1-s)^n}{n!} A^n \\
&= \sum_{k=0}^\infty \sum_{n=0}^\infty \frac{1}{k!n!} A^k B A^n \int_0^1 s^n (1-s)^k \,\mathrm{d}s \\
&= \sum_{k=0}^\infty \sum_{n=0}^\infty \frac{1}{k!n!} A^k B A^n \frac{k!n!}{(1+k+n)!} \\
&= \sum_{k=0}^\infty \sum_{n=0}^\infty \frac{1}{(1+k+n)!} A^k B A^n.
\end{aligned}$$

Here we have used

$$\int_0^1 s^n (1-s)^k \,\mathrm{d}s = \frac{k!n!}{(1+k+n)!},$$

a property of the Beta function.

It is easy to see that the two double series coincide (they have the same terms). Throughout the calculation we have used that all series/integrals converge absolutely, allowing us to exchange order of summation. □