Karlsruhe Institute of Technology (KIT) SS 2010
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Boundary and Eigenvalue Problems: 1st problem sheet

This exercise sheet is about boundary value problems for ordinary differential equations.

Exercise 1

i) Let $f \in C[0,1]$. Prove that the boundary value problem

$$-u'' = f \quad \text{in } (0,1) \qquad\qquad u(0) = u(1) = 0$$

has a unique solution given by $u(x) = \int_0^1 G(x,t) f(t)\, dt$ where

$$G(x,t) = \begin{cases} (1-x)t, & 0 \le t \le x \le 1\,, \\ (1-t)x, & 0 \le x \le t \le 1\,. \end{cases}$$

ii) Let $f \in C[0,\frac{\pi}{2}]$. Prove that the boundary value problem

$$-u'' - u = f \quad \text{in } (0,\frac{\pi}{2}) \qquad\qquad u(0) = u(\frac{\pi}{2}) = 0$$

has a unique solution given by $u(x) = \int_0^{\frac{\pi}{2}} G(x,t) f(t)\, dt$ where

$$G(x,t) = \begin{cases} \cos(x)\sin(t), & 0 \le t \le x \le \frac{\pi}{2}\,, \\ \sin(x)\cos(t), & 0 \le x \le t \le \frac{\pi}{2}\,. \end{cases}$$

Exercise 2

Determine all solutions of the eigenvalue problem

$$-u'' = \lambda u \qquad\qquad u(0) = u'(0),\; u(1) = u'(1)\,,$$

where $\lambda \in \mathbb{R}$ is given.

Exercise 3

Let $p \in C^1[0,1]$ be a positive function, $r \in C[0,1]$, $\eta_0, \eta_1 \in \mathbb{R}$. Show that the boundary value problem

$$-\frac{d}{dx}\big(p(x)u'(x)\big) = r(x) \qquad\qquad u'(0) = \eta_0,\ u'(1) = \eta_1$$

is solvable if and only if

$$(*) \qquad\qquad \int_0^1 r(x)\,dx = -p(1)\eta_1 + p(0)\eta_0\,.$$

In case $(*)$ holds find an explicit formula for the solution(s). Is the solution uniquely determined?