Karlsruhe Institute of Technology (KIT) SS 2010
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Boundary and Eigenvalue Problems:
10th problem sheet

Exercise 1

Determine all eigenvalues $\lambda \in \mathbb{R}$ and eigenvectors of the boundary value problem

$$\begin{aligned} -x^2 u''(x) &= \lambda u(x)\,, \qquad x \in (1, e)\,, \\ u(1) = u(e) &= 0\,. \end{aligned}$$

Exercise 2

Let

$$T : l^2 \to l^2,\ (x_1, x_2, \ldots,) \mapsto (x_2, x_3, \ldots)$$

be the shift operator on l^2. Determine $\sigma_p(T), \sigma(T), \sigma_p(T^*), \sigma(T^*)$.

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Consider the following problem: Given $f \in L^2(\Omega)$ find $u \in W^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u \nabla \phi \, dx = \int_\Omega f \phi \, dx \qquad \forall \phi \in W^{1,2}(\Omega)\,. \tag{$*$}$$

a) Explain why $(*)$ can be considered as the weak formulation of the Neumann boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}$$

b) Show that $(*)$ is solvable if and only if $\int_\Omega f(x)\,dx = 0$. The following hints may be useful:

i) Proceed as in the case of the Dirichlet boundary value problem: Find a compact operator $K : L^2(\Omega) \to L^2(\Omega)$ such that $(*)$ is equivalent to solving $(\mathrm{Id} - K)u = Kf$.

ii) In i) you may use the compactness of the Sobolev imbedding $W^{1,2}(\Omega) \to L^2(\Omega)$.

iii) $\int_\Omega |\nabla u|^2\,dx = 0$ for $u \in W^{1,2}(\Omega)$ implies that $u \equiv \mathrm{const}$ a.e. in Ω.