

**Boundary and Eigenvalue Problems:
11th problem sheet**

Exercise 1

a) Let $u \in C^2(\mathbb{R}^3)$ be given and let $\tilde{u} \in C^2((0, \infty) \times [0, 2\pi) \times [0, \pi))$ be defined by

$$\tilde{u}(r, \theta, t) = u(r \cos \theta \cos t, r \cos \theta \sin t, r \sin \theta).$$

Prove for all $(x, y, z) = (r \cos \theta \cos t, r \cos \theta \sin t, r \sin \theta)$ that

$$\Delta u(x, y, z) = \left(\frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{2}{r} \frac{\partial \tilde{u}}{\partial r} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \tilde{u}}{\partial t^2} + \frac{1}{r^2} \frac{\partial^2 \tilde{u}}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial \tilde{u}}{\partial \theta} \right)(r, \theta, t).$$

b) Let $\tilde{u} \in C^2(0, \infty)$. Show

$$\Delta(\tilde{u}(r)) = \tilde{u}''(r) + \frac{n-1}{r} \tilde{u}'(r), \quad r = |x|.$$

Exercise 2

Let $s \geq 0, s \notin \mathbb{N}_0$ and $B_R := \{x \in \mathbb{R}^2 : |x| < R\}$. Prove that every solution w of Bessel's differential equation

$$w'' + \frac{1}{r} w' + \left(1 - \frac{s^2}{r^2}\right) w = 0 \quad \text{in } (0, \infty)$$

that satisfies one of the following conditions

i) $w(|\cdot|) \in W^{1,2}(B_R)$

ii) $w(|\cdot|) \in C^2(B_R) \cap C(\overline{B_R})$

is a constant multiple of J_s .

Exercise 3

Let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Determine all radially symmetric solutions $u \in C^2(B_R) \cap C(\overline{B_R})$ of the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } B_R(0) \\ u &= 0 && \text{on } \partial B_R(0). \end{aligned}$$

Try $w(r) := r^{\frac{n-2}{2}} u(x)$ for $r = |x|$.