Karlsruhe Institute of Technology (KIT) SS 2010
Institut für Analysis
Prof. Dr. Wolfgang Reichel
Dipl.-Math.techn. Rainer Mandel

Boundary and Eigenvalue Problems: 11th problem sheet

Exercise 1

a) Let $u \in C^2(\mathbb{R}^3)$ be given and let $\tilde{u} \in C^2((0,\infty) \times [0,2\pi) \times [0,\pi))$ be defined by

$$\tilde{u}(r,\theta,t) = u(r\sin\theta\cos t, r\sin\theta\sin t, r\cos\theta)\,.$$

Prove for all $(x,y,z) = (r\sin\theta\cos t, r\sin\theta\sin t, r\cos\theta)$ that

$$\Delta u(x,y,z) = \Big(\frac{\partial^2\tilde{u}}{\partial r^2} + \frac{2}{r}\frac{\partial\tilde{u}}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\tilde{u}}{\partial t^2} + \frac{1}{r^2}\frac{\partial^2\tilde{u}}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\tilde{u}}{\partial\theta}\Big)(r,\theta,t)\,.$$

b) Let $\tilde{u} \in C^2(0,\infty)$. Show

$$\Delta(\tilde{u}(r)) = \tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r)\,, \qquad r = |x|\,.$$

Exercise 2

Let $s \geq 0, s \notin \mathbb{N}_0$ and $B_R := \{x \in \mathbb{R}^2 : |x| < R\}$. Prove that every solution w of Bessel's differential equation

$$w'' + \frac{1}{r}w' + (1 - \frac{s^2}{r^2})w = 0 \quad \text{in } (0,\infty)$$

that satisfies one of the following conditions

i) $w(|\cdot|) \in W^{1,2}(B_R)$

ii) $w(|\cdot|) \in C^2(B_R) \cap C(\overline{B}_R)$

is a constant multiple of J_s.

Exercise 3

Let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Determine all radially symmetric solutions $u \in C^2(B_R) \cap C(\overline{B}_R)$ of the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda u \quad && \text{in } B_R(0) \\ u &= 0 && \text{on } \partial B_R(0)\,. \end{aligned}$$

Try $w(r) := r^{\frac{n-2}{2}} u(x)$ for $r = |x|$.