Karlsruhe Institute of Technology (KIT) SS 2010
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Boundary and Eigenvalue Problems: 12th problem sheet

Exercise 1

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\varphi_1, \varphi_2, \ldots$ be an L^2-ONB of Dirichlet eigenfunctions of $-\Delta$. Let $\lambda \in \mathbb{R}$. Find suitable conditions on $f \in L^2(\Omega)$ and determine $\alpha_i \in \mathbb{R}$ such that the solution $u \in W_0^{1,2}(\Omega)$ of the eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u + f \quad &\text{in } \Omega, \\ u &= 0 &\text{on } \partial\Omega \end{aligned}$$

is given by $u = \sum_{i=1}^{\infty} \alpha_i \varphi_i$.

Exercise 2

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\lambda_1, \lambda_2, \ldots$ be the Dirichlet eigenvalues of $-\Delta$ with a corresponding L^2-ONB of eigenfunctions $\varphi_1, \varphi_2, \ldots$. Let $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

i) Show that there are coefficients $\alpha_i \in \mathbb{R}$ such that

$$u = \sum_{i=1}^{\infty} \alpha_i \varphi_i, \qquad -\Delta u = \sum_{i=1}^{\infty} \lambda_i \alpha_i \varphi_i.$$

ii) Let $\mu \in \mathbb{R}$ and δ be given by

$$\delta = \frac{\|\Delta u + \mu u\|_2}{\|u\|_2}.$$

Show that the interval $[\mu - \delta, \mu + \delta]$ contains at least one Dirichlet eigenvalue λ_i.

Exercise 3

For $R > 0$ let $\Omega = \{x \in \mathbb{R}^3 : R < |x| < 2R\}$ be the open annulus with radii $R, 2R$ in $\mathbb{R}^3$. Determine all radially symmetric Dirichlet eigenfunctions of the boundary value problem

$$\begin{aligned} -\Delta u = \lambda u & \quad \text{in } B_R(0) \\ u = 0 & \quad \text{on } \partial B_R(0) \end{aligned}$$

and calculate the corresponding eigenvalues.
Try $w(r) := ru(x)$ for $r = |x|$.