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Boundary and Eigenvalue Problems: 13th problem sheet

On this exercise sheet you can use that every eigenfunction $v \in W_0^{1,2}(\Omega)$ of $-\Delta$ on a bounded Lipschitz domain Ω automatically lies in $C^\infty(\Omega) \cap C(\overline{\Omega})$.

Exercise 1

Let

$$E := \left\{(x,y) \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1\right\}, \qquad a > b,$$

be an ellipse. Find optimal enclosures $\lambda_1(E) \in [\lambda_1(\Omega_1), \lambda_1(\Omega_2)]$ for the first Dirichlet eigenvalue $\lambda_1(E)$ of $-\Delta$ on E for approximating sets $\Omega_2 \subset E \subset \Omega_1$ such that

a) Ω_1, Ω_2 are balls.

b) Ω_1, Ω_2 are rectangles.

Exercise 2

Let $\lambda_1 \leq \ldots \leq \lambda_n < \lambda_{n+1} \leq \ldots$ denote the Dirichlet eigenvalues of $-\Delta$ on a bounded Lipschitz domain Ω with a corresponding L^2-ONB of eigenfunctions $v_1, v_2, \ldots \in W_0^{1,2}(\Omega)$. We want to show that the open set $D := \{x \in \Omega : v_n(x) \neq 0\}$ has at most n open disjoint components.

a) Assume for contradiction $D = \bigcup_{j=1}^{n+1} D_j$ for pairwise disjoint open subsets D_j of D. Set $w_j := v_n \cdot 1_{D_j}$ for $j = 1, \ldots, n+1$. Show that every element of $\operatorname{span}\{w_1, \ldots, w_{n+1}\}$ has the same Rayleigh quotient as v_n.

b) Show that $\lambda_{n+1*}(W) \leq \lambda_n$ for every subspace W of $W_0^{1,2}(\Omega)$ with $\dim(W) = n$ and establish a contradiction.

Hint: In a) you may assume $w_j \in W_0^{1,2}(D_j)$.

Exercise 3

Let Ω be a bounded Lipschitz domain Ω and let λ_1 be the first Dirichlet eigenvalue of $-\Delta$ with eigenfunction $\varphi_1 \in W_0^{1,2}(\Omega)$. Our aim is to prove that λ_1 is a simple eigenvalue, i.e. its eigenspace has dimension one and that its only eigenfunction φ_1 has no zeros in Ω. In the following let

$$R(u) = \frac{\|\nabla u\|_2^2}{\|u\|_2^2}, \qquad u \in W_0^{1,2}(\Omega) \setminus \{0\},$$

denote the Rayleigh quotient of u.

a) Show that every minimizer $\tilde{u} \in W_0^{1,2}(\Omega)$ of the Rayleigh quotient, i.e. every $\tilde{u} \in W_0^{1,2}(\Omega)$ satisfying $R(\tilde{u}) = \lambda_1$, is an eigenfunction for $-\Delta$ with eigenvalue λ_1.

b) Conclude that if $\tilde{u}$ is an eigenfunction of $-\Delta$ for λ_1 then so is $|\tilde{u}|$.

c) Let $\tilde{u}$ be an eigenfunction of $-\Delta$ for λ_1. Apply the classical strong maximum principle to $|\tilde{u}|$ in order to show that $\tilde{u}$ has no zeros in Ω.

d) Show that there cannot exist two linearly independent eigenfunctions of $-\Delta$ for λ_1.

Hint: In a) use that the function $t \mapsto R(\tilde{u} + t\phi)$ has a local minimum at $t = 0$ for every $\phi \in W_0^{1,2}(\Omega)$.