Karlsruhe Institute of Technology (KIT) SS 2010
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Boundary and Eigenvalue Problems: 2nd problem sheet

Exercise 1

Prove that every rectangle $[0,a] \times [0,b] \subset \mathbb{R}^2$ $(a,b>0)$ is a Lipschitz domain, but not a C^1-domain.

Exercise 2

Find a solution u of the form $u(x,y) = v(x)w(y)$ for the boundary value problem

$$\begin{aligned} -\Delta u &= \frac{\pi^2}{4} u \qquad \text{in } [0,1]\times[0,1] \\ u(0,y) = \cos(\frac{\pi}{2}y),\ u(1,y) &= 0 \qquad (y \in [0,1]) \\ u(x,0) = 1-x,\ u(x,1) &= 0 \qquad (x \in [0,1])\,. \end{aligned}$$

Exercise 3

i) Let $\alpha > 1$. Prove that every α-Hölder continuous function $u : \Omega \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ a domain is constant.

ii) a) Let $\alpha \in (0,1]$. Prove that the function $x \mapsto |x|^\alpha$ is α-Hölder continuous in $\mathbb{R}^n$.
b) Let $0 < \beta \leq 1 < \alpha$. Show that the function $x \mapsto |x|^\alpha$ is not β-Hölder continuous on $\mathbb{R}^n$ but only locally β-Hölder continuous.

iii) Prove that if $\Omega \subset \mathbb{R}^n$ is bounded we have $C^{0,\beta}(\overline{\Omega}) \subset C^{0,\alpha}(\overline{\Omega})$ for all $0 < \alpha \leq \beta \leq 1$.

iv) Let $\Omega \subset \mathbb{R}^n$, $\alpha \in (0,1]$ and suppose $f : \Omega \to \mathbb{R}$ is an α-Hölder continuous function with $C_\alpha := [f]_{\alpha,\Omega}$. Prove that

$$F(x) = \inf_{y\in\Omega}\{f(y) + C_\alpha |x-y|^\alpha\} \qquad (x \in \mathbb{R}^n)$$

defines an α-Hölder continuous extension of f satisfying $[F]_{\alpha,\mathbb{R}^n} = C_\alpha$.

Hint: In iv) you may use $(1+t)^\alpha - 1 \leq t^\alpha$ for all $t > 0, \alpha \in [0,1]$.