Karlsruhe Institute of Technology (KIT) SS 2010
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Boundary and Eigenvalue Problems: 3rd problem sheet

Exercise 1

Let $B_1(0)$ be the unit ball in $\mathbb{R}^n$, $f \in C(\overline{B_1(0)}), g \in C(\partial B_1(0))$.

a) Prove that Green's function for $B_1(0)$ is strictly positive in $B_1(0)$.

b) Show that if $u \in C^2(\overline{B_1(0)})$ is a solution of the problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } B_1(0) \\ u &= g \qquad \text{on } \partial B_1(0) \end{aligned} \tag{$*$}$$

and $f, g \geq 0$ then $u \geq 0$.

c) Prove that problem $(*)$ has at most one solution in $C^2(\overline{B_1(0)})$.

Exercise 2

Use Poisson's integral formula (Corollary 14) to prove that every positive harmonic function $u \in C^2(B_R(0))$ satisfies the inequality

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}} u(0) \leq u(x) \leq \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}} u(0) \qquad \forall x \in B_R(0)\,.$$

Show that this inequality does not hold if the positivity assumption is dropped.

Exercise 3

In this exercise we are interested in solving the boundary value problem

$$\begin{aligned} -\Delta u(x) &= 0 && \text{in } \mathbb{R}^n \times (0,\infty) \\ \lim_{\substack{x \to (x',0) \\ x \in \mathbb{R}^n \times (0,\infty)}} u(x) &= g(x') && \text{on } \mathbb{R}^n \times \{0\} \end{aligned} \qquad (**)$$

for $g : \mathbb{R}^n \to \mathbb{R}$ continuous and bounded.

a) Prove that the function

$$u(x', x_{n+1}) = \frac{2}{\omega_{n+1}} \int_{\mathbb{R}^n} \frac{x_{n+1}}{(|x'-y'|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} g(y')\, dy'$$

defines a solution for $(**)$ where ω_n denotes the surface area of $\partial B_1(0)$. You may use

$$1 = \frac{2}{\omega_{n+1}} \int_{\mathbb{R}^n} \frac{x_{n+1}}{(|x'-y'|^2 + x_{n+1}^2)^{\frac{n+1}{2}}}\, dy' \quad \text{for all } x_{n+1} > 0, x' \in \mathbb{R}^n$$

b) Determine the Green's function for the ball $B_R(z) \subset \mathbb{R}^n$ for arbitrary $z \in \mathbb{R}^n, R > 0$.

c) Let Γ_R be the Green's function for $B_R(Re_n)$ where e_n denotes the n-th unit vector in $\mathbb{R}^n$. Prove

$$\Gamma_R(x,y) \to \gamma(x-y) - \gamma(x-\hat{y}) \quad \text{as } R \to \infty$$

where γ is the fundamental solution of $-\Delta$ and $\hat{y} = (y_1, \ldots, y_{n-1}, -y_n)$.

Supplementary exercise

Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be the fundamental solution of $-\Delta$ in $\mathbb{R}^n$ (see Lemma 4 of the lecture) and let ν_j denote the j-th component of the outer unit normal field ν on $\partial B_1(0)$.

a) Show that the following identity holds for $x \in B_1(0)$:

$$\oint_{\partial B_1(0)} \gamma(x-y)\nu_j\, ds_y = \int_{B_1(0)} -\frac{\partial}{\partial x_j}\gamma(x-y)\, dy\,.$$

b) Use a) to prove

$$\oint_{\partial B_1(0)} -\frac{\partial}{\partial x_i}\gamma(x-y)\nu_j\, ds_y = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\int_{B_1(0)} \gamma(x-y)\, dy\,.$$

Remark: This fact was used in the proof of Proposition 13.