Karlsruhe Institute of Technology (KIT) SS 2010
Institut für Analysis
Prof. Dr. Wolfgang Reichel
Dipl.-Math.techn. Rainer Mandel

Boundary and Eigenvalue Problems: 4th problem sheet

Exercise 1

Determine all points $x = (x_1, x_2) \in \mathbb{R}^2$ in which the differential operator

$$Lu = x_1 \frac{\partial^2 u}{\partial x_1^2} + 2x_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + x_1 \frac{\partial^2 u}{\partial x_2^2} + (x_1^2 + x_2^2) \frac{\partial u}{\partial x_1} + x_1^2 x_2 \frac{\partial u}{\partial x_2} + \sin(x_1 + x_2) u$$

is elliptic.

Exercise 2

Prove the Lemma of Féjer-Schur:
If $A, B \in \mathbb{R}^{n \times n}$ are symmetric positive semi-definite matrices then $\mathrm{tr}(AB) \geq 0$.

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{aligned} -\Delta u(x) + c(x)u(x) &\leq 0 \qquad \text{in } \Omega \\ u(x) &\leq 0 \qquad \text{on } \partial\Omega\,. \end{aligned}$$

Give examples of Ω, u, c and $n = 1, 2$ showing that in general we cannot conclude $u(x) \leq 0$ for all $x \in \Omega$ without assuming $c \geq 0$.

Exercise 4

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and L a uniformly elliptic operator given by

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

where a_{ij}, b_i, c are bounded functions.

a) Prove that if there exists a function $h \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $Lh > 0$ in Ω and $h > 0$ on $\overline{\Omega}$ then the following maximum principle holds:
If $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $Lu \leq 0$ in Ω, $u \leq 0$ on $\partial\Omega$ then $u \leq 0$ in Ω.

b) Assume $\Omega \subset \{x \in \mathbb{R}^n : 0 < x_1 < d\}$. Use a) to show that the above maximum principle holds on Ω provided d is sufficiently small.

Remark:
i) Note that in a) there is no sign condition on c.
ii) Consider $u - \mu_0 h$ *with* $\mu_0 = \inf\{\mu > 0 : u - \mu h \leq 0 \text{ in } \overline{\Omega}\}$ *and show by contradiction that* $\mu_0 = 0$.