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SS 2010

Boundary and Eigenvalue Problems: 5th problem sheet

Exercise 1

Let $\Omega := B_1(0) \cap \{x_n > 0\}$ be a half ball in $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of the boundary value problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \cap \{x_n = 0\} \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \setminus \{x_n = 0\}\,. \end{aligned}$$

Use the maximum principle/Hopf's lemma to prove that $u \equiv 0$.

Exercise 2

a) Let $U_1 := B_r(z)$ be a ball and $U_2 := \{x \in \mathbb{R}^n : a < x_1 < a + d\}$ be an infinite strip. Find the solutions v_i of the boundary value problems

$$\begin{aligned} -\Delta v_i(x) &= 1 \quad \text{in } U_i \\ v_i(x) &= 0 \quad \text{on } \partial U_i \end{aligned}$$

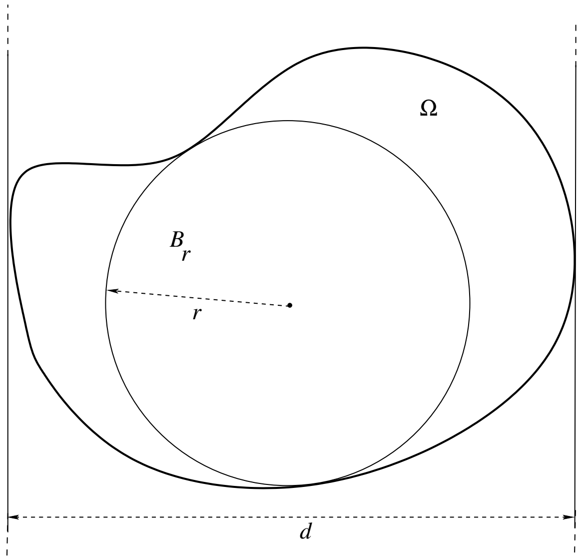
for $i = 1, 2$. In both cases try polynomials of second order.

b) Let Ω be a bounded domain such that $U_1 \subset \Omega \subset U_2$. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of the boundary value problem

$$\begin{aligned} -\Delta u(x) &= 1 \quad \text{in } \Omega \\ u(x) &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}$$

Use the maximum principle to show that $v_1 \le u$ on U_1 and $u \le v_2$ on Ω. Conclude

$$\frac{r^2}{2n} \le \max_{x \in \overline{\Omega}} u(x) \le \frac{d^2}{8}\,.$$



Exercise 3

Let $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ a solution of the uniformly elliptic differential equation

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u(x) = 0\,.$$

with coefficients $a_{ij} \in C^1(\Omega)$ such that $\|a_{ij}\|_\infty, \|\partial_k a_{ij}\|_\infty < \infty$. Show via the maximum principle that there exists a constant $C > 0$ independent of u such that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C(\|\nabla u\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)})$$

Hint: Consider $v := |\nabla u|^2 + t|u|^2$ for large $t > 0$ and compute Lv. Use $\partial_k Lu = 0$.