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# Boundary and Eigenvalue Problems: 8th problem sheet

## Exercise 1

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $k \in C(\overline{\Omega} \times \overline{\Omega})$. Prove that the operator

$$K : C(\overline{\Omega}, \|\cdot\|_\infty) \to C(\overline{\Omega}, \|\cdot\|_\infty), \quad u \mapsto (Ku)(x) := \int_\Omega k(x,y)u(y)\,dy$$

is compact.
*Hint:* Use the Arzelà-Ascoli theorem.

## Exercise 2

Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space and $S, T : E \to E$ be linear operators. Prove the following assertions or give a counterexample:

a) If $T$ is compact then $T$ is bounded.

b) If $S$ is bounded and $T$ is compact then $S \circ T$ and $T \circ S$ are compact.

c) If $T$ is compact then $T^2$ is compact.

d) If $T^2$ is compact then $T$ is compact.

e) If $\dim(\mathrm{Rg}(T)) < \infty$ then $T$ is compact.

f) If $\dim(\mathrm{Rg}(T)) < \infty$ and $T$ is bounded then $T$ is compact.

**Exercise 3**

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $a_{ij}, c \in L^\infty(\Omega)$ for all $i, j \in \{1, \dots, n\}$ and $b_i \in C^1(\overline{\Omega})$ satisfying

i) $2c(x) - \sum_{i=1}^n \partial_i b_i(x) \geq 0$ for all $x \in \overline{\Omega}$.

ii) $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for some $\lambda > 0$ for all $x \in \overline{\Omega}$.

Use the theorem of Lax-Milgram to prove that for all $f \in L^2(\Omega)$ the boundary value problem

$$-\sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j u(x)) + \sum_{i=1}^n b_i(x)\partial_i u(x) + c(x)u(x) = f(x) \quad \text{in } \Omega$$

$$u = 0 \qquad \text{on } \partial\Omega$$

has a unique weak solution $u \in W_0^{1,2}(\Omega)$.