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# Boundary and Eigenvalue Problems: 9th problem sheet

In the following let $l^2 = \{(x_i)_{i\in\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ be the normed space of square summable real sequences equipped with the inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$. You may use that $(l^2, \langle\cdot,\cdot\rangle)$ is a real Hilbert space.

## Exercise 1

Let $(a_{ij})_{i,j\in\mathbb{N}}$ be a double sequence such that $\sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty$ and such that $A_N = (a_{ij})_{1\le i,j\le N}$ is positive semidefinite for all $N \in \mathbb{N}$.

a) Prove that $T : l^2 \to l^2, v \mapsto (\sum_{i,j=1}^{\infty} a_{ij} v_j)_{i\in\mathbb{N}}$ is well-defined, linear and compact.

b) Use Fredholm's Alternative to prove that the equation

$$u_i + \sum_{j=1}^{\infty} a_{ij} u_j = f_i \qquad (i \in \mathbb{N})$$

is uniquely solvable for all $f \in l^2$.

*Hint for a):* Given a bounded sequence $(v^n) \in l^2$ use Cantor's diagonal argument to select a subsequence $(v^{n_j})$ such that every component sequence $(v_i^{n_j})_{j\in\mathbb{N}}$ converges.

## Exercise 3

Let $I := [\alpha, \beta] \subset \mathbb{R}$ be a bounded interval and $a \in C^1(I), c \in C(I)$ such that $a(x) \ge \lambda > 0$. Our aim is to derive Fredholm's alternative by elementary means for the ODE boundary value problem

$$\begin{aligned} -(a(x)u')' + c(x)u &= f(x) \qquad (x \in I)\,, \\ u'(\alpha) = u'(\beta) &= 0\,. \end{aligned} \tag{$*$}$$

a) Write down a solution formula of the differential equation in terms of a fundamental system $\{\xi, \eta\}$ of the homogeneous differential equation. Use variation of constants.

b) Prove that the boundary value problem $(*)$ is uniquely solvable for all $f \in C(I)$ if and only if $\int_I f(x)w(x)\,dx = 0$ for all $w$ satisfying

$$\begin{aligned} -(a(x)w')' + c(x)w &= 0 \qquad (x \in I)\,, \\ w'(\alpha) = w'(\beta) &= 0\,. \end{aligned}$$

Determine $\gamma \in \mathbb{R}$ such that the problem

$$\begin{aligned} u'' + u &= e^x - \gamma x \qquad (x \in (0,\pi))\,, \\ u'(0) = u'(\pi) &= 0 \end{aligned}$$

is solvable and calculate all solutions.

*Hint:* Use (and maybe prove) that the Wronskian $W(x) = \det \begin{pmatrix} \xi(x) & \eta(x) \\ \xi'(x) & \eta'(x) \end{pmatrix}$ of the fundamental system $\{\xi, \eta\}$ is constant.