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Boundary and Eigenvalue Problems – Summer Semester 2010
Handout on Functional Analysis/Hilbert Spaces

Ref.: W. Rudin, Real and Complex Analysis, 3^{rd} Ed., McGraw-Hill, 1987, Chapter 4.

Definition F.1 (Bounded, compact linear operators) *Let $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ be normed spaces. A linear operator $A : X \to Y$ is called*

(i) bounded, if
$$\|A\| := \sup_{x \in X \setminus \{0\}} \frac{|||Ax|||}{\|x\|} < \infty.$$
In the case $Y = \mathbb{R}$ bounded linear operators are called bounded linear **functionals***.*

(ii) compact, if for every bounded sequence $(x_k)_{k\in\mathbb{N}}$ in X the sequence $(Ax_k)_{k\in\mathbb{N}}$ in Y has a convergent subsequence.

Definition F.2 (Hilbert space) *Let H be a real vector space with inner product $\langle\cdot,\cdot\rangle$, i.e.*

(i) $\langle x, y\rangle = \langle y, x\rangle$,

(ii) $\langle x, x\rangle \geq 0$ *and* $= 0$ *if and only if* $x = 0$,

(iii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

Then
$$\|x\| := \sqrt{\langle x, x\rangle}$$
defines a norm on H. If H equipped with the above norm is complete, then $(H, \langle\cdot,\cdot\rangle)$ is called a Hilbert space.

Definition F.3 (Orthogonality) *Let $\langle\cdot,\cdot\rangle$ be an inner product on the real vector space H and let $V \subset H$.*

(i) $x \perp y :\Leftrightarrow \langle x, y\rangle = 0$,

(ii) $x \perp V :\Leftrightarrow x \perp v$ *for all* $v \in V$,

(iii) $V^\perp := \{x \in H : x \perp V\}$.

Theorem F.4 (Distance minimizer) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space and let* $V \subset H$ *be a closed subspace. Then*

(i) $\forall x \in H$ *there exists a unique* $v_0 \in V$ *such that*

$$\|x - v_0\| = \operatorname{dist}(x, V) = \inf_{v \in V} \|x - v\|.$$

Moreover $x - v_0 \perp V$*.*

(ii) $H = V \oplus V^{\perp}$*.*

Theorem F.5 (Riesz representation theorem) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a real Hilbert space and let* $\phi : H \to \mathbb{R}$ *be a bounded linear functional. Then there exists a unique* $u \in H$ *such that*

$$\phi(x) = \langle u, x\rangle \text{ for all } x \in H.$$

Note: $\|\phi\| = \|u\|$*.*

Definition F.6 (Weak convergence) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a real Hilbert space. A sequence* $(x_k)_{k\in\mathbb{N}}$ *in* H *is called weakly convergent to* $x \in H$ *if*

$$\lim_{k\to\infty} \langle x_k, y\rangle = \langle x, y\rangle \text{ for all } y \in H.$$

One writes $x_k \rightharpoonup x$ *as* $k \to \infty$ *for* **weakly convergent** *sequences. In contrast, one writes* $x_k \to x$ *as* $k \to \infty$ *for* **strongly convergent** *(norm convergent) sequences, i.e. if* $\|x_k - x\| \to 0$ *as* $k \to \infty$*.*

Lemma F.7 (Relation between weak and strong convergence) *Let* $(x_k)_{k\in\mathbb{N}}$ *be a sequence in a Hilbert space* H *and let* $x \in H$*.*

(i) If $x_k \to x$ *as* $k \to \infty$ *then* $x_k \rightharpoonup x$ *as* $k \to \infty$*. In infinite dimensions the reverse is in general false.*

(ii) The following equivalence holds:

$$x_k \to x \text{ as } k \to \infty \iff x_k \rightharpoonup x \text{ and } \|x_k\| \to \|x\| \text{ as } k \to \infty.$$

Theorem F.8 (Banach-Alaoglu for Hilbert spaces) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a real Hilbert space and let* $(x_k)_{k\in\mathbb{N}}$ *be a sequence in* H*.*

(i) If $(x_k)_{k\in\mathbb{N}}$ *weakly converges to* $x \in H$ *then the sequence* $(\|x_k\|)_{k\in\mathbb{N}}$ *is bounded and*

$$\|x\| \leq \liminf \|x_k\|.$$

(ii) If $(x_k)_{k\in\mathbb{N}}$ *is bounded then there exists a weakly convergent subsequence.*

Definition F.9 (Separable space) *A Banach space $(X, \|\cdot\|)$ is called separable, if there exists a countable set $Z = \{z_1, z_2, z_3, \ldots\}$, $Z \subset X$ such that $\overline{Z} = X$.*

Examples: $L^p(\Omega), W^{k,p}(\Omega)$ are separable if $1 \le p < \infty$. $L^\infty(\Omega), W^{k,\infty}(\Omega)$ are not separable if Ω is open and $\Omega \neq \emptyset$.

Definition F.10 (Orthonormal system, orthonormal basis) *Let $(H, \langle \cdot, \cdot \rangle)$ be a real, infinite-dimensional Hilbert space. A set $B = \{u_i : i \in \mathbb{N}\} \subset H$ is called an orthonormal system if*

$$\langle u_i, u_j \rangle = \delta_{ij}.$$

If additionally,

$$u = \sum_{i=1}^{\infty} \langle u, u_i \rangle u_i \quad \text{for all } u \in H,$$

then B is called an orthonormal basis.

Theorem F.11 (Existence of orthonormal basis) *Let $(H, \langle \cdot, \cdot \rangle)$ be a real, separable Hilbert space. Then H has an orthonormal basis.*

Theorem F.12 (Convergence of the abstract Fourier series) *Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and let $B = \{u_i : i \in \mathbb{N}\}$ be an orthonormal system. For every $u \in H$ the series*

$$\hat{u} := \sum_{i=1}^{\infty} \langle u, u_i \rangle u_i \quad \text{(abstract Fourier series)}$$

is convergent.

Proof: Fix $u \in H$. For $k \in \mathbb{N}$ let $\hat{u}_k := \sum_{i=1}^{k} \langle u, u_i \rangle u_i$. Then $\|\hat{u}_k\|^2 = \sum_{i=1}^{k} |\langle u, u_i \rangle|^2$ and

Bessel's equation:
$$\|u - \hat{u}_k\|^2 = \|u\|^2 - \sum_{i=1}^{k} |\langle u, u_i \rangle|^2 = \|u\|^2 - \|\hat{u}_k\|^2.$$

Hence

Bessel's inequality:
$$\sum_{i=1}^{\infty} |\langle u, u_i \rangle|^2 \le \|u\|^2.$$

For $k > l$

$$\|\hat{u}_k - \hat{u}_l\|^2 = \| \sum_{i=l+1}^{k} \langle u, u_i \rangle u_i \|^2 = \sum_{i=l+1}^{k} |\langle u, u_i \rangle|^2 < \epsilon \text{ for } k > l \ge l_0(\epsilon)$$

due to Bessel's inequality. Hence $(\hat{u}_k)_{k \in \mathbb{N}}$ is a Cauchy sequence and thus convergent. $\square$