

Boundary and Eigenvalue Problems – Summer Semester 2010

Handout on the Divergence Theorem

Definition G.1 An open, connected set $\Omega \subset \mathbb{R}^n$ is called a C^1 -domain (a Lipschitz-domain) if for every $x_0 \in \partial\Omega$ there exists a radius $r > 0$ and a C^1 -function (Lipschitz-function) $\gamma : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ such that (up to rotation of the coordinate frame)

$$\Omega \cap B_r(x_0) = \{x = (x', x_n) \in B_r(x_0) : x_n > \gamma(x')\}.$$

Note that as a consequence

$$\partial\Omega \cap B_r(x_0) = \{x = (x', x_n) \in B_r(x_0) : x_n = \gamma(x')\}.$$

For $x = (x', \gamma(x')) \in \partial\Omega \cap B_r(x_0)$ the vector

$$\nu(x) = \frac{(\nabla\gamma(x'), -1)}{\sqrt{1 + |\nabla\gamma(x')|^2}}$$

is called the exterior unit-normal of $\partial\Omega$ at x . In case of a Lipschitz-domain the exterior unit-normal ν exists a.e. with respect to the surface-measure on $\partial\Omega$.

Definition G.2 (Divergence theorem, Gauss theorem) Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 -domain and let ν be the exterior unit-normal on $\partial\Omega$. Then

$$\int_{\Omega} \frac{\partial f}{\partial x_i} dx = \oint_{\partial\Omega} f \nu_i d\sigma$$

for every function $f \in C^1(\overline{\Omega})$. The result also holds if Ω is a bounded Lipschitz-domain.

Often the divergence theorem appears in the following form

$$\int_{\Omega} \operatorname{div} F dx = \oint_{\partial\Omega} F \cdot \nu d\sigma$$

for a vectorfield $F : \overline{\Omega} \rightarrow \mathbb{R}^n$. The components of $F = (F_1, \dots, F_n)$ have to satisfy $F_i \in C^1(\overline{\Omega})$ for $i = 1, \dots, n$.

Lemma G.3 (Green's identities) Let $u, v \in C^2(\overline{\Omega})$. Then

$$\begin{aligned} \int_{\Omega} \Delta u dx &= \oint_{\partial\Omega} \nabla u \cdot \nu d\sigma, \\ \int_{\Omega} \nabla u \cdot \nabla v dx &= - \int_{\Omega} u \Delta v dx + \oint_{\partial\Omega} u \nabla v \cdot \nu d\sigma, \\ \int_{\Omega} (u \Delta v - v \Delta u) dx &= \oint_{\partial\Omega} (u \nabla v - v \nabla u) \cdot \nu d\sigma. \end{aligned}$$