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Boundary and Eigenvalue Problems – Summer Semester 2010
Handout on the Lebesgue Integral

Ref.: W. Rudin, Real and Complex Analysis, 3^{rd} Ed., McGraw-Hill, 1987, Chapter 1–3.

Let X be a set, e.g. $X = \mathbb{R}^n$, and let $\mathcal{P}(X) =$ be the set of all subsets of X.

Definition L.1 (σ-algebra) *A system of sets $\mathcal{M} \subset \mathcal{P}(X)$ is called a σ-algebra over X if*

(i) $X \in \mathcal{M}$

(ii) $A \in \mathcal{M} \Longrightarrow X \setminus A \in \mathcal{M}$

(iii) $A_i \in \mathcal{M} \quad \forall i \in \mathbb{N} \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$

Definition L.2 (positive measure) *Let $\mathcal{M}$ be σ-algebra over X. A map $\mu : \mathcal{M} \to [0, \infty]$ is called a positive measure, if*

$$A_i \in \mathcal{M} \quad \forall i \in \mathbb{N} \text{ with } A_i \cap A_j = \emptyset \text{ for } i \neq j \Longrightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Let $I = I^1 \times \cdots \times I^n = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be an open intervall in $\mathbb{R}^n$. Then the *volume* of I is defined by $|I| = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$. The same applies if one or more of the component-intervals (a_i, b_i) are replaced by closed, semi-closed, semi-open intervals.

Definition L.3 (outer measure on $\mathbb{R}^n$) *Let $A \subset \mathbb{R}^n$ be an arbitrary set. Then*

$$\lambda(A) := \inf\left\{\sum_{i=1}^{\infty} |I_i| : \quad A \subset \bigcup_{i=1}^{\infty} I_i \text{ und } I_i \text{ bounded interval } \forall i \in \mathbb{N}\right\}$$

is called the outer measure of the set A.

Remark: λ is not a positive measure on $\mathcal{P}(\mathbb{R}^n)$.

Definition L.4 (Lebesgue σ-algebra, Caratheodory) *A set $A \subset \mathbb{R}^n$ is called Lebesgue-measurable (short: $A \in \mathcal{L}(\mathbb{R}^n)$) if*

$$\lambda(E) = \lambda(A \cap E) + \lambda(A^c \cap E) \quad \forall E \subset \mathbb{R}^n.$$

If $X \subset \mathbb{R}^n$ is Lebesgue-measurable, then let $\mathcal{L}(X) = \{A \subset X : A \text{ is Lebesgue-measurable}\}$.

Theorem L.5 *$\mathcal{L}(\mathbb{R}^n)$ is a σ-algebra. The outer measure λ (see Definition L.3) is invariant under Eulcidean motions and if it is restricted to $\mathcal{L}(\mathbb{R}^n)$ then it becomes a positive, complete measure on $\mathcal{L}(\mathbb{R}^n)$.*

In the following let $X \subset \mathbb{R}^n$ be a Lebesgue-measurable set. For $f : X \to \overline{\mathbb{R}} = [-\infty, \infty]$ let $f^+ = \max\{f, 0\}, f^- = -\min\{f, 0\}$. Hence $f = f^+ - f^-$.

Definition L.6 (mesasurable functions)

(i) A function $f : X \to \overline{\mathbb{R}}$ is called measurable, if $f^{-1}\big((\alpha, \infty]\big) \in \mathcal{L}(X)$ for all $\alpha \in \mathbb{R}$,

(ii) A function $s : X \to \mathbb{R}$ is called an elementary function, if s possesses only finitely many values $\alpha_1, \ldots, \alpha_k$. In this case

$$s = \sum_{i=1}^{k} \alpha_i \chi_{A_i}, \quad A_i = s^{-1}(\alpha_i).$$

Definition L.7 (Lebesgue-integral for non-negative functions)

(i) Let $s = \sum_{i=1}^{k} \alpha_i \chi_{A_i}$ be a measurable elementary function. Then

$$\int_X s\,dx := \sum_{i=1}^{k} \alpha_i \lambda(A_i)$$

is called the Lebesgue-integral of s over X.

(ii) Let $f : X \to [0, \infty]$ be measurable. Then

$$\int_X f\,dx := \sup_{s \in \mathcal{S}} \int_X s\,dx, \quad \mathcal{S} = \{s : X \to \mathbb{R} \text{ measurable elementary function } , 0 \le s \le f\}$$

is called the Lebesgue-integral of f over the set X.

Definition L.8 (Lebesgue-integral for real- or complex-valued functions) *Let $I\!K = \overline{\mathbb{R}}$ or $\mathbb{C}$.*

$$L^1(X) := \{f : X \to I\!K \text{ measurable } : \int_X |f|\,dx < \infty\}.$$

For $f \in L^1(X)$ let $f_1 = \Re f, f_2 = \Im f$. Then

$$\int_X f\,dx := \int_X f_1^+\,dx - \int_X f_1^-\,dx + i\left(\int_X f_2^+\,dx - \int_X f_2^-\,dx\right)$$

is called the Lebesgue-integral of f over the set X.

Definition L.9 ($f = g$ a.e.) *Let $f, g : X \to I\!K$ be measurable. Then we say $f = g$ almost everywhere, if there exists a set N of measure 0 such that $f(x) = g(x)\ \forall x \in X \setminus N$. Equality almost everywhere is an equivalence relation.*

Definition L.10 (The space $L^p(X)$)

(a) For $1 \le p < \infty$ let

$$L^p(X) = \{u : X \to \overline{\mathbb{R}} \text{ measurable: } \int_X |u|^p \, dx < \infty\}.$$

(b) For $p = \infty$ let

$$L^\infty(X) = \{u : X \to \overline{\mathbb{R}} \text{ measurable: } \operatorname{ess\,sup}_X |u| < \infty\},$$

where $\operatorname{ess\,sup}_X v = \inf\{s \in \overline{\mathbb{R}} : v(x) \le s \text{ for almost all } x \in X\}$.

Definition L.11 (Norm on $L^p(X)$) *For $1 \le p < \infty$ let*

$$\|u\|_p := \left(\int_X |u|^p \, dx \right)^{1/p}$$

and

$$\|u\|_\infty := \operatorname{ess\,sup}_X |u|.$$

Then $(L^p(X), \| \cdot \|_p)$ is a Banach space.

Theorem L.12 (Minkowski and Hölder inequalities)

(i) $\|u + v\|_p \le \|u\|_p + \|v\|_p$ for all $u, v \in L^p(X)$.

(ii) Let $1 \le p, q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_X |uv| \, dx \le \|u\|_p \|v\|_q$$

for all $u \in L^p(X)$ and all $v \in L^q(X)$.

Theorem L.13 *Let $1 \le p \le \infty$ and $u \in L^p(X)$. If $(u_k)_{k \in \mathbb{N}}$ is a sequence of functions in $L^p(X)$ such that $\lim_{k \to \infty} \|u_k - u\|_p = 0$ then there exists a subsequence $(u_{k_l})_{l \in \mathbb{N}}$ such that*

$$\lim_{l \to \infty} u_{k_l}(x) = u(x) \text{ for almost all } x \in X.$$

Theorem L.14 (Monotone convergence) *Let $(u_k)_{k \in \mathbb{N}}$ be a sequence of measurable functions on X such that*

$$0 \le u_1 \le u_2 \le u_3 \le \dots .$$

Then $u(x) := \lim_{k \to \infty} u_k(x)$ exists for almost all $x \in X$ and

$$\lim_{k \to \infty} \int_X u_k \, dx = \int_X u \, dx.$$

Theorem L.15 (Dominated convergence) *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of measurable functions on X. If there exists $w \in L^1(X)$ such that $|u_k(x)| \leq w(x)$ for almost all $x \in X$ and all $k \in \mathbb{N}$ and if $u(x) := \lim_{k\to\infty} u_k(x)$ exists almost everywhere in X then*

$$\lim_{k\to\infty} \int_X u_k \, dx = \int_X u \, dx.$$

Theorem L.16 (Fatou's Lemma) *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of measurable functions on X such that $u_k(x) \geq 0$ almost everywhere on X. Then*

$$\int_X \liminf_{k\in\mathbb{N}} u_k \, dx \leq \liminf_{k\in\mathbb{N}} \int_X u_k \, dx.$$

Theorem L.17 *Let $1 \leq p < \infty$. Then the set of continuous functions with compact support $C_c(X)$ is dense in $L^p(X)$.*

Theorem L.18 (Dual space of $L^p(X)$) *Let $1 \leq p < \infty$ and let $\phi : L^p(X) \to \mathbb{R}$ be a continuous linear functional. Then there exists a unique $v \in L^q(X)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that*

$$\phi(u) = \int_X uv \, dx \textit{ for all } u \in L^p(X).$$

For short: $(L^p(X))^* = L^q(X)$.

Note: In general the theorem fails for $p = \infty$, i.e., $(L^\infty(X))^* \supsetneq L^1(X)$.