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Boundary and Eigenvalue Problems – Summer Semester 2010
Handout on Neumann boundary conditions

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let

$$L = -\sum_{i,j=1}^{n} \partial_i (a_{ij}(x)\partial_j) + \sum_{i=1}^{n} b_i(x)\partial_i + c(x)$$

be a uniformly elliptic operator in divergence form with bounded coefficients.

Theorem N.1 (First existence theorem) *There exists a value $\mu_0 \geq 0$ such that for all $\mu \geq \mu_0$ and all $f \in L^2(\Omega)$ there exists a unique weak solution $u \in W^{1,2}(\Omega)$ which solves the Neumann boundary value problem*

$$Lu + \mu u = f \text{ in } \Omega, \qquad \nu^T A(x)\nabla u = 0 \text{ on } \partial\Omega.$$

The linear operator

$$K : \begin{cases} L^2(\Omega) & \to & L^2(\Omega), \\ f & \mapsto & u, \end{cases}$$

which maps the right hand side f to the solution u is compact.

Theorem N.2 (Fredholm alternative) *Either*

(I) for every $f \in L^2(\Omega)$ the boundary value problem

$$(*) \qquad Lu = f \text{ in } \Omega, \qquad \nu^T A(x)\nabla u = 0 \text{ on } \partial\Omega$$

has a unique solution $u \in W^{1,2}(\Omega)$

or

(II) there is a nontrivial solution $u \in W^{1,2}(\Omega)$ of

$$Lu = 0 \text{ in } \Omega, \qquad \nu^T A(x)\nabla u = 0 \text{ on } \partial\Omega.$$

In case (II) we have additionally

$$(*) \text{ is solvable } \Leftrightarrow \langle f, v\rangle_{L^2} = 0 \text{ for every weak solution } v \in W^{1,2}(\Omega)$$

of the adjoint boundary value problem

$$L^* v = 0 \text{ in } \Omega, \qquad \nu^T A(x)\nabla v = 0 \text{ on } \partial\Omega.$$

Definition N.3 (Spectrum) *The Neumann-spectrum $\Sigma \subset \mathbb{R}$ ($\Sigma \subset \mathbb{C}$) of L is characterized as follows: $\mu \in \mathbb{R} \setminus \Sigma$ ($\mu \in \mathbb{C} \setminus \Sigma$) $\Leftrightarrow$ for every $f \in L^2(\Omega)$ the boundary value problem*

$$(*) \qquad Lu = f \text{ in } \Omega, \qquad \nu^T A(x) \nabla u = 0 \text{ on } \partial\Omega$$

has a unique solution $u \in W^{1,2}(\Omega)$.

Theorem N.4 (Properties of the spectrum) *The Neumann-spectrum of L has the following properties:*

(a) Σ is at most countable.

(b) If $\Sigma = \{\mu_k\}_{k \in \mathbb{N}}$ is an infinite sequence then $\mu_k \to \infty$ as $k \to \infty$.

(c) If $\mu \in \Sigma$ then μ is an eigenvalue, i.e., there exists a non-trivial function $\psi \in W^{1,2}(\Omega)$ which solves

$$L\psi = \mu\psi \text{ in } \Omega, \qquad \nu^T A(x) \nabla \psi = 0 \text{ on } \partial\Omega.$$

Theorem N.5 (The spectrum for symmetric operators) *If $L = L^*$ and $c(x) \geq 0$ then L has infinitely many eigenvalues $\Sigma = \{\mu_k\}_{k \in \mathbb{N}}$ (every eigenvalue is repeated according to its multiplicity) with*

$$0 \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots, \qquad \mu_k \to \infty \text{ as } k \to \infty$$

and there exists an orthonormal basis $\{\psi_k\}_{k \in \mathbb{N}}$ of $L^2(\Omega)$ consisting of eigenfunctions corresponding to μ_k.

In the following three theorems we assume that L is symmetric ($L = L^*$) and that $c(x) \geq 0$.

Theorem N.6 (Variational characterization of the spectrum)

(a) Let

$$\tilde{\mu}_1 = \inf\{B[u,u], u \in W^{1,2}(\Omega), \|u\|_{L^2} = 1\}.$$

Then $\tilde{\mu}_1$ is attained and $\tilde{\mu}_1 = \mu_1$.

(b) Let $i \geq 2$ and

$$\tilde{\mu}_i = \inf\{B[u,u], u \in W^{1,2}(\Omega), \|u\|_{L^2} = 1, \int_\Omega u\psi_k \, dx = 0, k = 1, \ldots, i-1\}.$$

Then $\tilde{\mu}_i$ is attained and $\tilde{\mu}_i = \mu_i$.

Theorem N.7 (Min-Max principle) *Let $i \geq 1$ and $W \subset W^{1,2}(\Omega)$ be a subspace with $\dim W = i$ and let*

$$\mu_i^*(W) = \max\{B[u,u], u \in W, \|u\|_{L^2} = 1\}.$$

Then we have

$$\mu_i = \min_{\dim W = i} \mu_i^*(W),$$

where the minimum is taken over all i-dimensional subspaces W of $W^{1,2}(\Omega)$.

Theorem N.8 (Max-Min principle) *Let $i \geq 2$ and $W \subset W^{1,2}(\Omega)$ be a subspace with $\dim W = i - 1$ and let*

$$\mu_{i,*}(W) = \min\{B[u,u], u \perp_{L^2} W, \|u\|_{L^2} = 1\}.$$

Then we have

$$\mu_i = \max_{\dim W = i-1} \mu_{i,*}(W),$$

where the maximum is taken over all $i - 1$-dimensional subspaces W of $W^{1,2}(\Omega)$.